

T-Run

[5, 5, 1, 5, 2], [3, 4, 5, 2, 1]

$$\tilde{\pi} = [2, 2, 1, 1, 3]$$

$$\delta = [2, 2, 1, 1, 4]$$

POSSIBLE RANKS

$$1 \times 9$$

$$3 \times 3$$

BASE DETERMINANT 351/4096, .8569335938e-1

NullSpace of Δ

{1, 2, 3, 4, 5}

Nullspace of A

$$\det(A) = -1/16$$

STRATIFIED CYCLE COVERS

Degree 0

1

Degree 1

0

Degree 2

$$v[2] v[4] + v[2] v[5] + v[1] v[3] + v[1] v[5]$$

Degree 3

$$v[2] v[4] v[5] + v[1] v[3] v[5]$$

Degree 4

$$v[1] v[2] v[4] v[5] + v[1] v[2] v[3] v[5] + v[1] v[2] v[3] v[4]$$

Degree 5

$$2 v[1] v[2] v[3] v[4] v[5]$$

{2}

$$R: [5, 4, 1, 5, 2]$$

$$B: [3, 5, 5, 2, 1]$$

TRACE TWO = 1

$$\det AT = \frac{3}{16} (-1 + t) (1 + 3t^2) (1 + t)$$

$$AT = \begin{pmatrix} 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{1}{6} & 0 & 0 & 0 & \frac{5}{6} \\ 0 & \frac{5}{6} & 0 & 0 & \frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} & 0 & 0 & 0 \end{pmatrix}$$

AT ranking is 3, "vs", 5

$$\text{Level 2 det} = \frac{-3}{65536} (3 + s) (-1 + s) (-52 - 55s - 4s^2 + 3s^3) (-36 - 13s + 12s^2 + s^3)$$

RANK of R is 4

R ranking is 2, "vs", 4

RBAR ranking 1, "vs", 3

RANK of B is 4

B ranking is 2, "vs", 4

BBAR ranking 1, "vs", 3

"R CYCLES", 1 + v[2] v[4] v[5]

"B CYCLES", 1 + v[1] v[3] v[5]

Eigenvalues

R: [0., 0., 1., -0.5000000000 + 0.8660254040 I, -0.5000000000 - 0.8660254040 I]

B: [0., 0., 1., -0.5000000000 + 0.8660254040 I, -0.5000000000 - 0.8660254040 I]

NullSpace of R

{[0, 0, 1, 0, 0]}

NullSpace of B

{[0, 0, 0, 1, 0]}

NullSpace of R^*

{[-1, 0, 0, 1, 0]}

NullSpace of B^*

{[0, -1, 1, 0, 0]}

FIXED POINTS DIMENSION 1

$$\text{PROTO-M} = \begin{pmatrix} 0 & 8 & 8 & 0 & 16 \\ 8 & 0 & 0 & 8 & 16 \\ 8 & 0 & 0 & 0 & 8 \\ 0 & 8 & 0 & 0 & 8 \\ 16 & 16 & 8 & 8 & 0 \end{pmatrix} \quad \text{N} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

"RANK of N is ", 3, "RANK of M is ", 5

"RANK of the KERNEL is ", 3

"IdemSolvability Check", 3 "Trace mark", 3, "Rank mark", 3, "for kernel rank", 3

degree 1: $\frac{1}{9} (2v[1] + 2v[2] + v[3] + v[4] + 3v[5])$

degree 2: $\frac{1}{9} (v[1]v[2] + v[1]v[3] + 2v[1]v[5] + v[2]v[4] + 2v[2]v[5] + v[3]v[5] + v[4]v[5])$

degree 3 : $\frac{1}{3} (v[5]) (v[1]v[2] + v[1]v[3] + v[2]v[4])$

Group spectrum $1 + t + t^2 + t^3$

KERNEL STRUCTURE

"PT1" = {{1, 4}, {5}, {2, 3}}

"RG1" = {2, 4, 5}

"RG2" = {1, 3, 5}

"RG3" = {1, 2, 5}

$$\pi_3 = [0, 0, 1, 0, 1, 0, 0, 0, 1, 0]$$

supp $\pi_3 = \{3, 5, 9\}$

$$u_3 = [0, 0, 1, 0, 1, 0, 0, 0, 1, 1]$$

supp $u_3 = \{3, 5, 9, 10\}$

Action of R on ranges, [[1], [3], [1]]

Action of B on ranges, [[3], [2], [2]]

$$\beta = \left(\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \right)$$

RPARTS [1]

BPARTS [1]

$$\alpha = (1)$$

Action of R and B on the blocks of the partitions:

R-BLOCKS,

[3, 1, 2]

B-BLOCKS,

[2, 3, 1]

with invariant measure, [1, 1, 1]

N by blocks, N - check: true

$b_1 = \{1, 4\}$

$b_2 = \{5\}$

$b_3 = \{2, 3\}$

dim(span of partition vectors), rank(N_0), rank(N): 3, 3, 3

LIE STRUCTURE

Dimension of Lie algebra: 11, Shape: $3 \oplus 8/6$

$$\text{CLB} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

R and B Cycles. V.

$$\Omega_R = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \Omega_B = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

Ω_R in Vec(K)?, {{2, 4, 5}}, true

Ω_B in Vec(K)?, {{1, 3, 5}}, true

$$V = \begin{pmatrix} \frac{-1}{9} & \frac{2}{9} & \frac{-5}{9} & \frac{1}{9} & \frac{1}{3} \\ \frac{-2}{9} & \frac{1}{9} & \frac{-1}{9} & \frac{5}{9} & \frac{-1}{3} \\ \frac{4}{9} & \frac{-2}{9} & \frac{2}{9} & \frac{-1}{9} & \frac{-1}{3} \\ \frac{2}{9} & \frac{-4}{9} & \frac{1}{9} & \frac{-2}{9} & \frac{1}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{-1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

Omega-checks true, true

$$\pi_R = \left(0 \ \frac{1}{3} \ 0 \ \frac{1}{3} \ \frac{1}{3}\right) \text{ vs } \left(0 \ \frac{1}{3} \ 0 \ \frac{1}{3} \ \frac{1}{3}\right) \quad u\Omega_R \text{ vs } \Omega(I-V)^{-1}$$

$$\pi_B = \left(\frac{1}{3} \ 0 \ \frac{1}{3} \ 0 \ \frac{1}{3}\right) \text{ vs } \left(\frac{1}{3} \ 0 \ \frac{1}{3} \ 0 \ \frac{1}{3}\right) \quad u\Omega_B \text{ vs } \Omega(I+V)^{-1}$$

LOCAL GROUPS

1, "partition", {{1, 4}, {5}, {2, 3}}

1, "range", [2, 4, 5], [[5, 4, 4, 5, 2], [4, 2, 2, 4, 5], [2, 5, 5, 2, 4]]

2, "range", [1, 3, 5], [[5, 1, 1, 5, 3], [3, 5, 5, 3, 1], [1, 3, 3, 1, 5]]

3, "range", [1, 2, 5], [[5, 1, 1, 5, 2], [2, 5, 5, 2, 1], [1, 2, 2, 1, 5]]

"group has", 3, "elements" Group element 1,1 =
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$g_1 = [[1, 2, 3]]$$

$$g_2 = []$$

$$g_3 = [[1, 3, 2]]$$

linear dimension, 3

"Symmetric?", false

Is Z in Vec(K)? true

$$(h[2] \ h[1] \ h[3])$$

"Basis for Z(G)"

1, "coeff", 1

$$Z[1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2, "coeff", 1

$$Z[2] = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

3, "coeff", 1

$$Z[3] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Check abelian

1, 2, true
1, 3, true
2, 3, true

$$\text{EIGS} = \begin{pmatrix} 1. & & 1. & & & & 1. \\ 1. & -0.5000000000 + 0.8660254040i & & -0.5000000000 - 0.8660254040i & & & \\ 1. & -0.5000000000 + 0.8660254040i & & -0.5000000000 - 0.8660254040i & & & \end{pmatrix}$$

PermChars :=

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Group spectrum: $1 + t + t^2 + t^3$

Molien Series to order 10: $1 + t + 2t^2 + 4t^3 + 5t^4 + 7t^5 + 10t^6 + 12t^7 + 15t^8 + 19t^9 + 22t^{10}$

n-choose-rank

{1, [1, 2, 3]}, {2, [1, 2, 4]}, {3, [1, 2, 5]}, {4, [1, 3, 4]}, {5, [1, 3, 5]}, {6, [1, 4, 5]}, {7, [2, 3, 4]}, {8, [2, 3, 5]},
{9, [2, 4, 5]}, {10, [3, 4, 5]}

KERNEL HIERARCHY

$\pi_3 = (0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0)$

{3, 5, 9}

$u_3 = (0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1)$

{3, 5, 9, 10}

picheck (2 2 1 1 3)

$\pi = \left(\frac{2}{9} \ \frac{2}{9} \ \frac{1}{9} \ \frac{1}{9} \ \frac{1}{3}\right)$

$\pi_2 = (1 \ 1 \ 0 \ 2 \ 0 \ 1 \ 2 \ 0 \ 1 \ 1)$

$u_2 = \left(\frac{1}{3} \ \frac{1}{3} \ 0 \ \frac{1}{3} \ 0 \ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3}\right)$

picheck (4 4 2 2 6)

$\pi_1 = (4 \ 4 \ 2 \ 2 \ 6)$

$u_1 = \left(\frac{2}{9} \ \frac{2}{9} \ \frac{2}{9} \ \frac{2}{9} \ \frac{2}{9}\right)$

picheck (4 4 2 2 6)

Column Projections

$$P_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks} \quad \text{NO-checks}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks} \quad \text{NO-checks}$$

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks} \quad \text{NO-checks}$$

Row Projections

$$PP_1 = \begin{pmatrix} \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks}$$

$$T = \begin{pmatrix} \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad NM = \begin{pmatrix} \frac{32}{9} & \frac{16}{9} & \frac{8}{9} & \frac{16}{9} & \frac{8}{3} \\ \frac{16}{9} & \frac{32}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{3} \\ \frac{16}{9} & \frac{32}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{3} \\ \frac{32}{9} & \frac{16}{9} & \frac{8}{9} & \frac{16}{9} & \frac{8}{3} \\ \frac{16}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{9} & \frac{16}{3} \end{pmatrix}$$

CHECKING NULLSPACES

$$\pi\Delta = [-1, 1, -1, 1, 0]$$

$$\ker N_c = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -t & t & 0 & 0 \\ -s & 0 & 0 & s & 0 \end{pmatrix} \text{ RB checks}$$

$\pi\Delta$ via $\ker NC (-1 \ 1)$

M_0 is invertible. $\det = 6500/19683$

$$\ker M_c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} t+s \\ t+s \\ t+s \\ t+s \\ t+s \end{pmatrix} \text{ RB checks}$$

$$n\pi x^\dagger = (5)$$

$$RN_0R^* = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, RR^* = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$BN_0B^* = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, BB^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Do these reconstitute N ? , true

Ranks: 4, 4, "vs", 3

$$CNM = \begin{pmatrix} 0 & 0 & \frac{-8}{9} & -1 & \frac{8}{9} \\ 0 & 0 & -1 & \frac{-8}{9} & \frac{8}{9} \\ \frac{8}{9} & 1 & 0 & 0 & \frac{16}{9} \\ 1 & \frac{8}{9} & 0 & 0 & \frac{16}{9} \\ \frac{-8}{9} & \frac{-8}{9} & \frac{-16}{9} & \frac{-16}{9} & 0 \end{pmatrix} \text{ Skew T} = \begin{pmatrix} 0 & 0 & 0 & \frac{-1}{3} & 0 \\ 0 & 0 & \frac{-1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ Skew Omega} =$$

$$\begin{pmatrix} 0 & 0 & \frac{-1}{9} & \frac{-1}{9} & \frac{1}{9} \\ 0 & 0 & \frac{-1}{9} & \frac{-1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & 0 & 0 & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & 0 & 0 & \frac{2}{9} \\ \frac{-1}{9} & \frac{-1}{9} & \frac{-2}{9} & \frac{-2}{9} & 0 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 2 & \frac{8}{9} & \frac{8}{9} & 0 & \frac{16}{9} \\ \frac{8}{9} & 2 & 0 & \frac{8}{9} & \frac{16}{9} \\ \frac{8}{9} & 0 & 1 & 0 & \frac{8}{9} \\ 0 & \frac{8}{9} & 0 & 1 & \frac{8}{9} \\ \frac{16}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{9} & 3 \end{pmatrix} \quad N_0 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

"IS NM a combination of T and Omega?", true

$$NM = \frac{8}{3} T + 8\Omega$$

$$\Omega \left(\frac{1}{9} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{2}{9} \quad \frac{1}{3} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{2}{9} \quad \frac{2}{9} \right)$$

$$\tau \left(0 \quad 0 \quad \frac{1}{3} \quad 0 \quad 0 \quad \frac{1}{3} \quad 0 \quad 0 \quad \frac{2}{3} \right)$$

"IS NM in Vec(K)?", true

$$NM \left(\frac{8}{9} \quad \frac{8}{9} \quad \frac{16}{9} \quad \frac{16}{9} \quad \frac{8}{3} \quad \frac{16}{9} \quad \frac{8}{9} \quad \frac{16}{9} \quad \frac{32}{9} \right)$$

"IS MN in Vec(K)?", false

$$MN \left(0 \quad 0 \quad 0 \quad \frac{8}{7} \quad \frac{16}{7} \quad \frac{8}{3} \quad \frac{8}{7} \quad \frac{8}{7} \quad \frac{8}{3} \right)$$

$$\tau = 9/1, \text{ rank} = 3, \text{ ratio} = 3/1, n^2 / r = 25/3$$

$$\tau' = 16/1, r' = 2/3, \tau / n^2 = 9/25$$

$$p^2 = 19/81, \text{ min } \tau = 475/81, \tau\text{-check is positive? } 254/81$$

$$\text{max } r = 81/19, r\text{-check is positive? } 8/27$$

IS N0M0 a combination of T and Omega? , true

$$N_0 M_0 = \frac{1}{3} T + 8\Omega$$

There are, 1, partitions and, 3, ranges, with a group size of, 3

KERNEL HAS LINEAR DIMENSION 9
out of total no. of elements equal to 9

dim span idems 3 vs no. of idems 3

"PT1" = {{1, 4}, {5}, {2, 3}}

"RG1" = {2, 4, 5}

"RG2" = {1, 3, 5}

"RG3" = {1, 2, 5}

$$M_C = \begin{pmatrix} \frac{62}{81} & \frac{-28}{81} & \frac{22}{81} & \frac{-50}{81} & \frac{-2}{27} \\ \frac{-28}{81} & \frac{62}{81} & \frac{-50}{81} & \frac{22}{81} & \frac{-2}{27} \\ \frac{22}{81} & \frac{-50}{81} & \frac{56}{81} & \frac{-25}{81} & \frac{-1}{27} \\ \frac{-50}{81} & \frac{22}{81} & \frac{-25}{81} & \frac{56}{81} & \frac{-1}{27} \\ \frac{-2}{27} & \frac{-2}{27} & \frac{-1}{27} & \frac{-1}{27} & \frac{2}{9} \end{pmatrix} \quad N_C = \begin{pmatrix} \frac{62}{81} & \frac{-19}{81} & \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} \\ \frac{-19}{81} & \frac{62}{81} & \frac{62}{81} & \frac{-19}{81} & \frac{-19}{81} \\ \frac{-19}{81} & \frac{62}{81} & \frac{62}{81} & \frac{-19}{81} & \frac{-19}{81} \\ \frac{62}{81} & \frac{-19}{81} & \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} \\ \frac{-19}{81} & \frac{-19}{81} & \frac{-19}{81} & \frac{-19}{81} & \frac{62}{81} \end{pmatrix}$$

$$M_C\text{-scaled} = \begin{pmatrix} 1 & \frac{-14}{31} & \frac{11}{31} & \frac{-25}{31} & \frac{-3}{31} \\ \frac{-14}{31} & 1 & \frac{-25}{31} & \frac{11}{31} & \frac{-3}{31} \\ \frac{11}{28} & \frac{-25}{28} & 1 & \frac{-25}{56} & \frac{-3}{56} \\ \frac{-25}{28} & \frac{11}{28} & \frac{-25}{56} & 1 & \frac{-3}{56} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{6} & \frac{-1}{6} & 1 \end{pmatrix} \quad N_C\text{-scaled} = \begin{pmatrix} 1 & \frac{-19}{62} & \frac{-19}{62} & 1 & \frac{-19}{62} \\ \frac{-19}{62} & 1 & 1 & \frac{-19}{62} & \frac{-19}{62} \\ \frac{-19}{62} & 1 & 1 & \frac{-19}{62} & \frac{-19}{62} \\ 1 & \frac{-19}{62} & \frac{-19}{62} & 1 & \frac{-19}{62} \\ \frac{-19}{62} & \frac{-19}{62} & \frac{-19}{62} & \frac{-19}{62} & 1 \end{pmatrix}$$

$$N_C M_C = \begin{pmatrix} \frac{4}{27} & \frac{-2}{27} & \frac{-1}{27} & \frac{2}{27} & \frac{-1}{9} \\ \frac{-2}{27} & \frac{4}{27} & \frac{2}{27} & \frac{-1}{27} & \frac{-1}{9} \\ \frac{-2}{27} & \frac{4}{27} & \frac{2}{27} & \frac{-1}{27} & \frac{-1}{9} \\ \frac{4}{27} & \frac{-2}{27} & \frac{-1}{27} & \frac{2}{27} & \frac{-1}{9} \\ \frac{-2}{27} & \frac{-2}{27} & \frac{-1}{27} & \frac{-1}{27} & \frac{2}{9} \end{pmatrix} \quad M_C N_C = \begin{pmatrix} \frac{4}{27} & \frac{-2}{27} & \frac{-2}{27} & \frac{4}{27} & \frac{-2}{27} \\ \frac{-2}{27} & \frac{4}{27} & \frac{4}{27} & \frac{-2}{27} & \frac{-2}{27} \\ \frac{-1}{27} & \frac{2}{27} & \frac{2}{27} & \frac{-1}{27} & \frac{-1}{27} \\ \frac{2}{27} & \frac{-1}{27} & \frac{-1}{27} & \frac{2}{27} & \frac{-1}{27} \\ \frac{-1}{9} & \frac{-1}{9} & \frac{-1}{9} & \frac{-1}{9} & \frac{2}{9} \end{pmatrix} \quad \text{commutator} =$$

$$\begin{pmatrix} 0 & 0 & \frac{-1}{27} & \frac{2}{27} & \frac{1}{27} \\ 0 & 0 & \frac{2}{27} & \frac{-1}{27} & \frac{1}{27} \\ \frac{1}{27} & \frac{-2}{27} & 0 & 0 & \frac{2}{27} \\ \frac{-2}{27} & \frac{1}{27} & 0 & 0 & \frac{2}{27} \\ \frac{-1}{27} & \frac{-1}{27} & \frac{-2}{27} & \frac{-2}{27} & 0 \end{pmatrix}$$

Eigenvalues M_C

[0., 1.946178864, 0.1649322482, 0.7505405108, 0.2741508472]

Eigenvalues N_C

[0., 0., 2., 1.405552070, 0.4216084235]

Eigenvalues M_C -scaled

[0., 1.135874376, 0.9660841503, 2.670943107, 0.227098367]

Eigenvalues N_C -scaled

[0., 0., 2.612903226, 1.836285769, 0.5508110047]

NullSpace M_C

{[1, 1, 1, 1, 1]}

NullSpace N_C

{[0, -1, 1, 0, 0], [-1, 0, 0, 1, 0]}

Eigenvalues M_0

[1.946178864, 0.1649322482, 5.929100987, 0.2416082767, 0.7181796227]

Eigenvalues N_0

[1., 0., 0., 2., 2.]

NullSpace M_0

{}

NullSpace N_0

{[-1, 0, 0, 1, 0], [0, 1, -1, 0, 0]}

Eigenvalues M

[0.5493635453, -1.438252434, 3.615324520, -2.195546624, -0.5308890059]

Eigenvalues N

[0., 0., -2., 3.236067977, -1.236067977]

NullSpace M

{}

NullSpace N

{[1, 0, 0, -1, 0], [0, 1, -1, 0, 0]}

Harmonic Basis

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

=====

{2, 3, 4, 5}

R: [5, 4, 5, 2, 1]
 B: [3, 5, 1, 5, 2]

TRACE TWO = 1

$$\det AT = \frac{3}{16} (-1 + t) (1 + 3t^2) (1 + t)$$

$$AT = \begin{pmatrix} 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{5}{6} & 0 & 0 & 0 & \frac{1}{6} \\ 0 & \frac{1}{6} & 0 & 0 & \frac{5}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 \end{pmatrix}$$

AT ranking is 3, "vs", 5

$$\text{Level 2 det} = \frac{-3}{65536} (-36 - 13s + 12s^2 + s^3) (-52 + 57s - 20s^2 + 3s^3) (3 + s) (-1 + s)$$

RANK of R is 4

R ranking is 2, "vs", 4

RBAR ranking 2, "vs", 4

RANK of B is 4

B ranking is 2, "vs", 4

BBAR ranking 2, "vs", 4

"R CYCLES", (1 + v[1] v[5]) (1 + v[2] v[4])

"B CYCLES", (1 + v[2] v[5]) (1 + v[1] v[3])

Eigenvalues

R: [0., 1., -1., 1., -1.]

B: [0., 1., -1., 1., -1.]

NullSpace of R

{[0, 0, 1, 0, 0]}

NullSpace of B

{[0, 0, 0, 1, 0]}

NullSpace of R*

{[-1, 0, 1, 0, 0]}

NullSpace of B*

{[0, 1, 0, -1, 0]}

FIXED POINTS DIMENSION 1

$$\text{PROTO-M} = \begin{pmatrix} 0 & 8 & 0 & 8 & 16 \\ 8 & 0 & 8 & 0 & 16 \\ 0 & 8 & 0 & 0 & 8 \\ 8 & 0 & 0 & 0 & 8 \\ 16 & 16 & 8 & 8 & 0 \end{pmatrix} \quad \text{N} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

"RANK of N is ", 3, "RANK of M is ", 5

"RANK of the KERNEL is ", 3

"IdemSolvability Check", 3 "Trace mark", 3, "Rank mark", 3, "for kernel rank", 3

degree 1: $\frac{1}{9} (2 v[1] + 2 v[2] + v[3] + v[4] + 3 v[5])$

degree 2: $\frac{1}{9} (v[1]v[2] + v[1]v[4] + 2 v[1]v[5] + v[2]v[3] + 2 v[2]v[5] + v[3]v[5] + v[4]v[5])$

degree 3 : $\frac{1}{3} (v[5]) (v[1]v[2] + v[1]v[4] + v[2]v[3])$

Group spectrum $1 + t + t^2 + t^3$

KERNEL STRUCTURE

"PT1" = {{1, 3}, {5}, {2, 4}}

"RG1" = {2, 3, 5}

"RG2" = {1, 4, 5}

"RG3" = {1, 2, 5}

$$\pi_3 = [0, 0, 1, 0, 0, 1, 0, 1, 0, 0]$$

supp $\pi_3 = \{3, 6, 8\}$

$$u_3 = [0, 0, 1, 0, 0, 1, 0, 1, 0, 1]$$

supp $u_3 = \{3, 6, 8, 10\}$

Action of R on ranges, [[2], [3], [2]]

Action of B on ranges, [[3], [1], [1]]

$$\beta = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

RPARTS [1]

BPARTS [1]

$$\alpha = (1)$$

Action of R and B on the blocks of the partitions:

R-BLOCKS,

[2, 1, 3]

B-BLOCKS,

[1, 3, 2]

with invariant measure, [1, 1, 1]

N by blocks, N - check: true

$$b_1 = \{1, 3\}$$

$$b_2 = \{5\}$$

$$b_3 = \{2, 4\}$$

dim(span of partition vectors), rank(N_0), rank(N): 3, 3, 3

LIE STRUCTURE

Dimension of Lie algebra: 11, Shape: $3 \oplus 8/6$

$$CLB = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

R and B Cycles. V.

$$\Omega_R = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \Omega_B = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

Ω_R in Vec(K)? , {{1, 5}, {2, 4}}, true

Ω_B in Vec(K)? , {{1, 3}, {2, 5}}, true

$$V = \begin{pmatrix} -\frac{1}{9} & \frac{2}{9} & -\frac{5}{9} & \frac{1}{9} & \frac{1}{3} \\ -\frac{2}{9} & \frac{1}{9} & -\frac{1}{9} & \frac{5}{9} & -\frac{1}{3} \\ -\frac{4}{9} & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} & \frac{1}{3} \\ -\frac{2}{9} & \frac{4}{9} & -\frac{1}{9} & \frac{2}{9} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \end{pmatrix}$$

Omega-checks true, true

"I-V is singular"

"I+V is singular"

$$\pi_R = \left(\frac{3}{10} \ \frac{1}{5} \ 0 \ \frac{1}{5} \ \frac{3}{10}\right) \text{ vs } (0 \ 0 \ 0 \ 0 \ 0) \quad u\Omega_R \text{ vs } \Omega(I-V)^{-1}$$

$$\pi_B = \left(\frac{1}{5} \ \frac{3}{10} \ \frac{1}{5} \ 0 \ \frac{3}{10}\right) \text{ vs } (0 \ 0 \ 0 \ 0 \ 0) \quad u\Omega_B \text{ vs } \Omega(I+V)^{-1}$$

LOCAL GROUPS

1, "partition", {{1, 3}, {5}, {2, 4}}

1, "range", [2, 3, 5], [[5, 3, 5, 3, 2], [5, 2, 5, 2, 3], [3, 5, 3, 5, 2], [3, 2, 3, 2, 5], [2, 5, 2, 5, 3], [2, 3, 2, 3, 5]]

2, "range", [1, 4, 5], [[5, 4, 5, 4, 1], [5, 1, 5, 1, 4], [4, 5, 4, 5, 1], [4, 1, 4, 1, 5], [1, 5, 1, 5, 4], [1, 4, 1, 4, 5]]

3, "range", [1, 2, 5], [[5, 2, 5, 2, 1], [5, 1, 5, 1, 2], [2, 5, 2, 5, 1], [2, 1, 2, 1, 5], [1, 5, 1, 5, 2], [1, 2, 1, 2, 5]]

"group has", 6, "elements" Group element 1,1 = $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$g_1 = [[1, 2, 3]]$$

$$g_2 = [[2, 3]]$$

$$g_3 = [[1, 3]]$$

$$g_4 = []$$

$$g_5 = [[1, 3, 2]]$$

linear dimension, 5

"Symmetric?", true

Is Z in Vec(K)? true

$$(h[2] \ 0 \ 0 \ 2h[1] \ h[2])$$

"Basis for Z(G)"

1, "coeff", 2

$$Z[1] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2, "coeff", 1

$$Z[2] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Check abelian

1, 2, true

$$\text{EIGS} = \begin{pmatrix} 1. & 1. & 1. \\ 2. & -1. & -1. \end{pmatrix}$$

PermChars :=

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 3 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Group spectrum: $1 + t + t^2 + t^3$

Molien Series to order 10: $1 + t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 7t^6 + 8t^7 + 10t^8 + 12t^9 + 14t^{10}$

n-choose-rank

{1, [1, 2, 3]}, {2, [1, 2, 4]}, {3, [1, 2, 5]}, {4, [1, 3, 4]}, {5, [1, 3, 5]}, {6, [1, 4, 5]}, {7, [2, 3, 4]}, {8, [2, 3, 5]}, {9, [2, 4, 5]}, {10, [3, 4, 5]}

KERNEL HIERARCHY

$$\pi_3 = (0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0)$$

{3, 6, 8}

$$\nu_3 = (0 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1)$$

{3, 6, 8, 10}

picheck (2 2 1 1 3)

$$\pi = \left(\frac{2}{9} \ \frac{2}{9} \ \frac{1}{9} \ \frac{1}{9} \ \frac{1}{3} \right)$$

$$\pi_2 = (1 \ 0 \ 1 \ 2 \ 1 \ 0 \ 2 \ 0 \ 1 \ 1)$$

$$u_2 = \left(\frac{1}{3} \ 0 \ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \ 0 \ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \right)$$

picheck (4 4 2 2 6)

$$\pi_1 = (4 \ 4 \ 2 \ 2 \ 6)$$

$$u_1 = \left(\frac{2}{9} \ \frac{2}{9} \ \frac{2}{9} \ \frac{2}{9} \ \frac{2}{9} \right)$$

picheck (4 4 2 2 6)

Column Projections

$$P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks} \quad \text{NO-checks}$$

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks} \quad \text{NO-checks}$$

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks} \quad \text{NO-checks}$$

Row Projections

$$PP_1 = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks}$$

$$T = \begin{pmatrix} \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad NM = \begin{pmatrix} \frac{32}{9} & \frac{16}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{3} \\ \frac{16}{9} & \frac{32}{9} & \frac{8}{9} & \frac{16}{9} & \frac{8}{3} \\ \frac{32}{9} & \frac{16}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{3} \\ \frac{16}{9} & \frac{32}{9} & \frac{8}{9} & \frac{16}{9} & \frac{8}{3} \\ \frac{16}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{9} & \frac{16}{3} \end{pmatrix}$$

CHECKING NULLSPACES

$$\pi\Delta = [1, -1, -1, 1, 0]$$

$$\ker N_C = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -t & 0 & t & 0 & 0 \\ 0 & -s & 0 & s & 0 \end{pmatrix} \quad \text{RB checks}$$

$\pi\Delta$ via ker NC (1 -1)

M0 is invertible. det= 6500/19683

$$\ker M_C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} t+s \\ t+s \\ t+s \\ t+s \\ t+s \end{pmatrix} \quad \text{RB checks}$$

$$n\pi x^\dagger = (5)$$

$$RN_0R^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad RR^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$BN_0B^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad BB^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Do these reconstitute N? , true

Ranks: 4, 4, "vs", 3

$$\text{CNM} = \begin{pmatrix} 0 & 0 & -1 & \frac{-8}{9} & \frac{8}{9} \\ 0 & 0 & \frac{-8}{9} & -1 & \frac{8}{9} \\ 1 & \frac{8}{9} & 0 & 0 & \frac{16}{9} \\ \frac{8}{9} & 1 & 0 & 0 & \frac{16}{9} \\ \frac{-8}{9} & \frac{-8}{9} & \frac{-16}{9} & \frac{-16}{9} & 0 \end{pmatrix} \quad \text{Skew T} = \begin{pmatrix} 0 & 0 & \frac{-1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{3} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Skew Omega} =$$

$$\begin{pmatrix} 0 & 0 & \frac{-1}{9} & \frac{-1}{9} & \frac{1}{9} \\ 0 & 0 & \frac{-1}{9} & \frac{-1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & 0 & 0 & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & 0 & 0 & \frac{2}{9} \\ \frac{-1}{9} & \frac{-1}{9} & \frac{-2}{9} & \frac{-2}{9} & 0 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 2 & \frac{8}{9} & 0 & \frac{8}{9} & \frac{16}{9} \\ \frac{8}{9} & 2 & \frac{8}{9} & 0 & \frac{16}{9} \\ 0 & \frac{8}{9} & 1 & 0 & \frac{8}{9} \\ \frac{8}{9} & 0 & 0 & 1 & \frac{8}{9} \\ \frac{16}{9} & \frac{16}{9} & \frac{8}{9} & \frac{8}{9} & 3 \end{pmatrix} \quad N_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

"IS NM a combination of T and Omega?", true

$$\text{NM} = \frac{8}{3}T + 8\Omega$$

$$\Omega \left(\frac{1}{9} \quad \frac{2}{9} \quad \frac{1}{3} \quad \frac{1}{9} \quad \frac{2}{9} \quad \frac{2}{9} \quad \frac{1}{3} \quad \frac{1}{9} \quad \frac{1}{9} \quad \frac{2}{9} \quad \frac{2}{9} \right)$$

$$T \left(0 \quad \frac{2}{3} \quad 1 \quad 0 \quad \frac{2}{3} \quad 0 \quad 0 \quad 0 \quad \frac{1}{3} \quad 0 \quad \frac{2}{3} \right)$$

"IS NM in Vec(K)?", true

$$\text{NM} \left(\frac{8}{9} \quad \frac{32}{9} \quad \frac{16}{3} \quad \frac{8}{9} \quad \frac{32}{9} \quad \frac{16}{9} \quad \frac{8}{3} \quad \frac{8}{9} \quad \frac{16}{9} \quad \frac{16}{9} \quad \frac{32}{9} \right)$$

"IS MN in Vec(K)?", false

$$\text{MN} \left(\frac{-2}{11} \quad \frac{92}{33} \quad \frac{184}{33} \quad \frac{14}{11} \quad \frac{86}{33} \quad \frac{14}{11} \quad \frac{24}{11} \quad \frac{14}{11} \quad \frac{86}{33} \quad \frac{14}{11} \quad \frac{86}{33} \right)$$

$$\tau = 9/1, \text{rank} = 3, \text{ratio} = 3/1, n^2/r = 25/3$$

$$\tau' = 16/1, r' = 2/3, \tau/n^2 = 9/25$$

$$p^2 = 19/81, \text{min } \tau = 475/81, \tau\text{-check is positive? } 254/81$$

$$\text{max } r = 81/19, r\text{-check is positive? } 8/27$$

IS N0M0 a combination of T and Omega? , true

$$N_0M_0 = \frac{1}{3}T + 8\Omega$$

There are, 1, partitions and, 3, ranges, with a group size of, 6

KERNEL HAS LINEAR DIMENSION 11
out of total no. of elements equal to 18

dim span idems 3 vs no. of idems 3

"PT1" = {{1, 3}, {5}, {2, 4}}

"RG1" = {2, 3, 5}

"RG2" = {1, 4, 5}

"RG3" = {1, 2, 5}

$$M_C = \begin{pmatrix} \frac{62}{81} & \frac{-28}{81} & \frac{-50}{81} & \frac{22}{81} & \frac{-2}{27} \\ \frac{-28}{81} & \frac{62}{81} & \frac{22}{81} & \frac{-50}{81} & \frac{-2}{27} \\ \frac{-50}{81} & \frac{22}{81} & \frac{56}{81} & \frac{-25}{81} & \frac{-1}{27} \\ \frac{22}{81} & \frac{-50}{81} & \frac{-25}{81} & \frac{56}{81} & \frac{-1}{27} \\ \frac{-2}{27} & \frac{-2}{27} & \frac{-1}{27} & \frac{-1}{27} & \frac{2}{9} \end{pmatrix} \quad N_C = \begin{pmatrix} \frac{62}{81} & \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} & \frac{-19}{81} \\ \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} \\ \frac{62}{81} & \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} & \frac{-19}{81} \\ \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} & \frac{62}{81} & \frac{-19}{81} \\ \frac{-19}{81} & \frac{-19}{81} & \frac{-19}{81} & \frac{-19}{81} & \frac{62}{81} \end{pmatrix}$$

$$M_C\text{-scaled} = \begin{pmatrix} 1 & \frac{-14}{31} & \frac{-25}{31} & \frac{11}{31} & \frac{-3}{31} \\ \frac{-14}{31} & 1 & \frac{11}{31} & \frac{-25}{31} & \frac{-3}{31} \\ \frac{-25}{28} & \frac{11}{28} & 1 & \frac{-25}{56} & \frac{-3}{56} \\ \frac{11}{28} & \frac{-25}{28} & \frac{-25}{56} & 1 & \frac{-3}{56} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{6} & \frac{-1}{6} & 1 \end{pmatrix} \quad N_C\text{-scaled} = \begin{pmatrix} 1 & \frac{-19}{62} & 1 & \frac{-19}{62} & \frac{-19}{62} \\ \frac{-19}{62} & 1 & \frac{-19}{62} & 1 & \frac{-19}{62} \\ 1 & \frac{-19}{62} & 1 & \frac{-19}{62} & \frac{-19}{62} \\ \frac{-19}{62} & 1 & \frac{-19}{62} & 1 & \frac{-19}{62} \\ \frac{-19}{62} & \frac{-19}{62} & \frac{-19}{62} & \frac{-19}{62} & 1 \end{pmatrix}$$

$$N_c M_c = \begin{pmatrix} \frac{4}{27} & \frac{-2}{27} & \frac{2}{27} & \frac{-1}{27} & \frac{-1}{9} \\ \frac{-2}{27} & \frac{4}{27} & \frac{-1}{27} & \frac{2}{27} & \frac{-1}{9} \\ \frac{4}{27} & \frac{-2}{27} & \frac{2}{27} & \frac{-1}{27} & \frac{-1}{9} \\ \frac{-2}{27} & \frac{4}{27} & \frac{-1}{27} & \frac{2}{27} & \frac{-1}{9} \\ \frac{-2}{27} & \frac{-2}{27} & \frac{-1}{27} & \frac{-1}{27} & \frac{2}{9} \end{pmatrix} \quad M_c N_c = \begin{pmatrix} \frac{4}{27} & \frac{-2}{27} & \frac{4}{27} & \frac{-2}{27} & \frac{-2}{27} \\ \frac{-2}{27} & \frac{4}{27} & \frac{-2}{27} & \frac{4}{27} & \frac{-2}{27} \\ \frac{2}{27} & \frac{-1}{27} & \frac{2}{27} & \frac{-1}{27} & \frac{-1}{27} \\ \frac{-1}{27} & \frac{2}{27} & \frac{-1}{27} & \frac{2}{27} & \frac{-1}{27} \\ \frac{-1}{9} & \frac{-1}{9} & \frac{-1}{9} & \frac{-1}{9} & \frac{2}{9} \end{pmatrix} \quad \text{commutator} =$$

$$\begin{pmatrix} 0 & 0 & \frac{2}{27} & \frac{-1}{27} & \frac{1}{27} \\ 0 & 0 & \frac{-1}{27} & \frac{2}{27} & \frac{1}{27} \\ \frac{-2}{27} & \frac{1}{27} & 0 & 0 & \frac{2}{27} \\ \frac{1}{27} & \frac{-2}{27} & 0 & 0 & \frac{2}{27} \\ \frac{-1}{27} & \frac{-1}{27} & \frac{-2}{27} & \frac{-2}{27} & 0 \end{pmatrix}$$

Eigenvalues M_c

[0., 1.946178864, 0.1649322482, 0.7505405108, 0.2741508472]

Eigenvalues N_c

[0., 0., 2., 1.405552070, 0.4216084235]

Eigenvalues M_c -scaled

[0., 1.135874376, 0.9660841503, 2.670943107, 0.227098367]

Eigenvalues N_c -scaled

[0., 0., 2.612903226, 1.836285769, 0.5508110047]

NullSpace M_c

{{[1, 1, 1, 1, 1]}}

NullSpace N_c

{{[0, -1, 0, 1, 0], [-1, 0, 1, 0, 0]}}

Eigenvalues M_0

[1.946178864, 0.1649322482, 5.929100987, 0.2416082767, 0.7181796227]

Eigenvalues N_0

[1., 0., 0., 2., 2.]

NullSpace M_0

{}

NullSpace N_0

{[0, 1, 0, -1, 0], [-1, 0, 1, 0, 0]}

Eigenvalues M

[0.5493635453, -1.438252434, 3.615324520, -2.195546624, -0.5308890059]

Eigenvalues N

[0., 0., -2., 3.236067977, -1.236067977]

NullSpace M

{}

NullSpace N

{[0, -1, 0, 1, 0], [-1, 0, 1, 0, 0]}

Harmonic Basis

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$