T - Run

[3, 4, 5, 5, 1], [2, 3, 1, 1, 4]

$\tilde{\pi}$ = [12, 6, 9, 7, 8]
$\boldsymbol{\delta}$ = [3, 1, 2, 2, 2]

POSSIBLE RANKS

1 x 42
2 x 21
3 x 14
6 x 7

BASE DETERMINANT 211/1024, .2060546875

NullSpace of Δ

{1, 2, 3, 4, 5}

Nullspace of A

[{1, 3},{2, 4, 5}]

STRATIFIED CYCLE COVERS

Degree 0
1

Degree 1
0

Degree 2
v[5] v[4] + v[1] v[3]

Degree 3
v[1] v[2] v[3] + v[1] v[3] v[5] + v[1] v[2] v[4]

Degree 4
v[1] v[2] v[3] v[5] + v[1] v[2] v[5] v[4] + 2 v[1] v[3] v[5] v[4]

Degree 5
2 v[1] v[2] v[3] v[5] v[4]

==

{3, 5}

R: [3, 4, 1, 5, 4]
B: [2, 3, 5, 1, 1]

TRACE TWO = 1

det AT = $\frac{-1}{8}(-1+t)^2(t)(1+t)$

$$AT = \begin{pmatrix} 0 & \frac{5}{6} & \frac{1}{6} & 0 & 0 \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} & 0 \\ \frac{1}{6} & 0 & 0 & 0 & \frac{5}{6} \\ \frac{5}{6} & 0 & 0 & 0 & \frac{1}{6} \\ \frac{5}{6} & 0 & 0 & \frac{1}{6} & 0 \end{pmatrix}$$

AT ranking is 4, "vs", 5

Level 2 det = $\frac{3}{8192}(-1688+468s+203s^2+25s^3-19s^4+3s^5)(-1+s)$

RANK of R is 4

R ranking is 2, "vs", 4

RBAR ranking 2, "vs", 4

RANK of B is 4

B ranking is 3, "vs", 4

BBAR ranking 3, "vs", 4

"R CYCLES", (1 + v[1] v[3]) (1 + v[5] v[4])
"B CYCLES", 1 + v[1] v[2] v[3] v[5]

Eigenvalues

R: [0., 1., −1., 1., −1.]

B: [0., −1., 1., 1. I, −1. I]

NullSpace of R

{[0, 1, 0, 0, 0]}

NullSpace of B

{[0, 0, 0, 1, 0]}

NullSpace of R^*

{[0, −1, 0, 0, 1]}

NullSpace of B^*

{[0, 0, 0, −1, 1]}

FIXED POINTS DIMENSION 1

$$\text{PROTO-M} = \begin{pmatrix} 0 & 8 & 0 & 4 & 12 \\ 8 & 0 & 4 & 0 & 0 \\ 0 & 4 & 0 & 10 & 4 \\ 4 & 0 & 10 & 0 & 0 \\ 12 & 0 & 4 & 0 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

"RANK of N is ", 2, "RANK of M is ", 4

"RANK of the KERNEL is ", 2

"IdemSolvability Check", 3 "Trace mark", 3, "Rank mark", 3, "for kernel rank", 2

degree 1: $\frac{4}{21}$ ($12v[1] + 6v[2] + 9v[3] + 7v[4] + 8v[5]$)
degree 2: $\frac{2}{21}$ ($4v[1]v[2] + 2v[1]v[4] + 6v[1]v[5] + 2v[2]v[3] + 5v[3]v[4] + 2v[3]v[5]$)

Group spectrum $1 + t + t^2$

KERNEL STRUCTURE

"PT1" = {{1, 3}, {2, 4, 5}}

"RG1" = {3, 5}

"RG2" = {3, 4}

"RG3" = {2, 3}

"RG4" = {1, 5}

"RG5" = {1, 4}

"RG6" = {1, 2}

$$\pi 2 = [4, 0, 2, 6, 2, 0, 0, 5, 2, 0]$$

supp $\pi 2$ = {1, 3, 4, 5, 8, 9}

$$u2 = [1, 0, 1, 1, 1, 0, 0, 1, 1, 0]$$

supp u2 = {1, 3, 4, 5, 8, 9}

Action of R on ranges, [[5], [4], [5], [2], [1], [2]]
Action of B on ranges, [[4], [4], [1], [6], [6], [3]]

$\boldsymbol{\beta} = \begin{pmatrix} \frac{2}{21} & \frac{5}{21} & \frac{2}{21} & \frac{2}{7} & \frac{2}{21} & \frac{4}{21} \end{pmatrix}$

RPARTS [1]

BPARTS [1]

$\boldsymbol{\alpha} = (1)$

Action of R and B on the blocks of the partitions:

R-BLOCKS,

[1, 2]

B−BLOCKS,

[2, 1]

with invariant measure, [1, 1]

N by blocks, N − check: true

$b_1 = \{1, 3\}$
$b_2 = \{2, 4, 5\}$

dim(span of partition vectors), rank(N_0), rank(N):   2, 2, 2

LIE STRUCTURE

Dimension of Lie algebra: 11, Shape: 3 ⊕ 8/6

$$\mathrm{CLB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

R and B Cycles. V.

$$\Omega_R = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \Omega_B = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$$

Ω_R in Vec(K)? , {{4, 5}, {1, 3}}, true

Ω_B in Vec(K)? , {{1, 2, 3, 5}}, true

$$\mathrm{V} = \begin{pmatrix} \frac{1}{7} & \frac{-3}{7} & \frac{5}{14} & \frac{-1}{6} & \frac{2}{21} \\ 0 & 0 & \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{2}{7} & \frac{1}{7} & \frac{3}{14} & \frac{-1}{6} & \frac{-10}{21} \\ \frac{-2}{7} & \frac{-1}{7} & \frac{-3}{14} & \frac{1}{6} & \frac{10}{21} \\ \frac{-2}{7} & \frac{-1}{7} & \frac{-3}{14} & \frac{1}{2} & \frac{1}{7} \end{pmatrix}$$

Omega−checks true, true

"I−V is singular"

$\pi_R = \begin{pmatrix} \frac{1}{5} & 0 & \frac{1}{5} & \frac{3}{10} & \frac{3}{10} \end{pmatrix}$ vs $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $u\Omega_R$ vs $\Omega(I-V)^{-1}$

$\pi_B = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$ vs $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix}$ $u\Omega_B$ vs $\Omega(I+V)^{-1}$

LOCAL GROUPS

1, "partition", {{1, 3}, {2, 4, 5}}

1, "range", [3, 5], [[5, 3, 5, 3, 3], [3, 5, 3, 5, 5]]

2, "range", [3, 4], [[4, 3, 4, 3, 3], [3, 4, 3, 4, 4]]

3, "range", [2, 3], [[3, 2, 3, 2, 2], [2, 3, 2, 3, 3]]

4, "range", [1, 5], [[5, 1, 5, 1, 1], [1, 5, 1, 5, 5]]

5, "range", [1, 4], [[4, 1, 4, 1, 1], [1, 4, 1, 4, 4]]

6, "range", [1, 2], [[2, 1, 2, 1, 1], [1, 2, 1, 2, 2]]

"group has", 2, "elements" Group element 1,1 = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$g_1 =$ [[1, 2]]

$g_2 =$ []

linear dimension, 2

"Symmetric?", true

Is Z in Vec(K)? true
$\begin{pmatrix} h[2] & h[1] \end{pmatrix}$

"Basis for Z(G)"

1, "coeff", 1

Z[1] = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2, "coeff", 1

Z[2] = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Check abelian

1, 2, true

EIGS = $\begin{pmatrix} 1. & 1. \\ 1. & -1. \end{pmatrix}$

PermChars :=

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

Group spectrum: $1 + t + t^2$

Molien Series to order 10: $1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + 5t^9 + 6t^{10}$

n-choose-rank

{1, [1, 2]}, {2, [1, 3]}, {3, [1, 4]}, {4, [1, 5]}, {5, [2, 3]}, {6, [2, 4]}, {7, [2, 5]}, {8, [3, 4]}, {9, [3, 5]}, {10, [4, 5]}

KERNEL HIERARCHY

$\pi 2 = \begin{pmatrix} 4 & 0 & 2 & 6 & 2 & 0 & 0 & 5 & 2 & 0 \end{pmatrix}$

{1, 3, 4, 5, 8, 9}

$u2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

{1, 3, 4, 5, 8, 9}

picheck $\begin{pmatrix} 12 & 6 & 9 & 7 & 8 \end{pmatrix}$

$\pi = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} & \frac{3}{14} & \frac{1}{6} & \frac{4}{21} \end{pmatrix}$

$\pi 1 = \begin{pmatrix} 12 & 6 & 9 & 7 & 8 \end{pmatrix}$

$u1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

picheck $\begin{pmatrix} 12 & 6 & 9 & 7 & 8 \end{pmatrix}$

Column Projections

$P_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ idem-checks N0-checks

$P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ idem-checks N0-checks

$$P_3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$ idem−checks N0−checks

$$P_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ idem−checks N0−checks

$$P_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$ idem−checks N0−checks

$$P_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$ idem−checks N0−checks

Row Projections

$$PP_1 = \begin{pmatrix} \frac{4}{7} & 0 & \frac{3}{7} & 0 & 0 \\ 0 & \frac{2}{7} & 0 & \frac{1}{3} & \frac{8}{21} \\ \frac{4}{7} & 0 & \frac{3}{7} & 0 & 0 \\ 0 & \frac{2}{7} & 0 & \frac{1}{3} & \frac{8}{21} \\ 0 & \frac{2}{7} & 0 & \frac{1}{3} & \frac{8}{21} \end{pmatrix}$$ idem−checks

$$T = \begin{pmatrix} \frac{4}{7} & 0 & \frac{3}{7} & 0 & 0 \\ 0 & \frac{2}{7} & 0 & \frac{1}{3} & \frac{8}{21} \\ \frac{4}{7} & 0 & \frac{3}{7} & 0 & 0 \\ 0 & \frac{2}{7} & 0 & \frac{1}{3} & \frac{8}{21} \\ 0 & \frac{2}{7} & 0 & \frac{1}{3} & \frac{8}{21} \end{pmatrix} \quad NM = \begin{pmatrix} \frac{24}{7} & 0 & \frac{18}{7} & 0 & 0 \\ 0 & \frac{12}{7} & 0 & 2 & \frac{16}{7} \\ \frac{24}{7} & 0 & \frac{18}{7} & 0 & 0 \\ 0 & \frac{12}{7} & 0 & 2 & \frac{16}{7} \\ 0 & \frac{12}{7} & 0 & 2 & \frac{16}{7} \end{pmatrix}$$

CHECKING NULLSPACES

$\pi\boldsymbol{\Delta}$ = [−3, −6, 3, 7, −1]

$$\ker N_C = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -s & t & s & 0 & -t \\ t & 0 & -t & -s & s \\ t & 0 & -t & 0 & 0 \end{pmatrix} \quad \text{RB checks}$$

$\pi\boldsymbol{\Delta}$ via ker NC $\begin{pmatrix} -3 & 7 & -1 \end{pmatrix}$

M0 is invertible. det= 281450/21609

$$\ker M_C = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} t+s \\ t+s \\ t+s \\ t+s \\ t+s \end{pmatrix} \quad \text{RB checks}$$

$n\pi x^{\dagger} = (5)$

$$\mathrm{R}N_0\mathrm{R}^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \mathrm{RR}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{B}N_0\mathrm{B}^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \mathrm{BB}^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Do these reconstitute N? , true

Ranks: 4, 4, "vs", 2

$$\mathrm{CNM} = \begin{pmatrix} 0 & \frac{-12}{7} & \frac{-13}{14} & \frac{-10}{7} & \frac{-8}{7} \\ \frac{12}{7} & 0 & \frac{6}{7} & \frac{13}{42} & \frac{13}{21} \\ \frac{13}{14} & \frac{-6}{7} & 0 & \frac{-4}{7} & \frac{-2}{7} \\ \frac{10}{7} & \frac{-13}{42} & \frac{4}{7} & 0 & \frac{13}{42} \\ \frac{8}{7} & \frac{-13}{21} & \frac{2}{7} & \frac{-13}{42} & 0 \end{pmatrix} \quad \text{Skew T} = \begin{pmatrix} 0 & 0 & \frac{-1}{7} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{21} & \frac{2}{21} \\ \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{21} & 0 & 0 & \frac{1}{21} \\ 0 & \frac{-2}{21} & 0 & \frac{-1}{21} & 0 \end{pmatrix} \quad \text{Skew Omega} =$$

$$\begin{pmatrix} 0 & \frac{-1}{7} & \frac{-1}{14} & \frac{-5}{42} & \frac{-2}{21} \\ \frac{1}{7} & 0 & \frac{1}{14} & \frac{1}{42} & \frac{1}{21} \\ \frac{1}{14} & \frac{-1}{14} & 0 & \frac{-1}{21} & \frac{-1}{42} \\ \frac{5}{42} & \frac{-1}{42} & \frac{1}{21} & 0 & \frac{1}{42} \\ \frac{2}{21} & \frac{-1}{21} & \frac{1}{42} & \frac{-1}{42} & 0 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} \frac{26}{7} & \frac{8}{7} & 0 & \frac{4}{7} & \frac{12}{7} \\ \frac{8}{7} & \frac{13}{7} & \frac{4}{7} & 0 & 0 \\ 0 & \frac{4}{7} & \frac{39}{14} & \frac{10}{7} & \frac{4}{7} \\ \frac{4}{7} & 0 & \frac{10}{7} & \frac{13}{6} & 0 \\ \frac{12}{7} & 0 & \frac{4}{7} & 0 & \frac{52}{21} \end{pmatrix} \quad N_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

"IS NM a combination of T and Omega?", true

$$\mathrm{NM} = 6T + 0\Omega$$

$\Omega \left(\frac{1}{6} \ \ \frac{1}{7} \ \ \frac{2}{7} \ \ \frac{4}{21} \ \ \frac{1}{6} \ \ \frac{3}{14} \ \ \frac{1}{7} \ \ \frac{2}{7} \right)$

$T \left(\frac{1}{3} \ \ \frac{2}{7} \ \ 0 \ \ 0 \ \ 0 \ \ \frac{3}{7} \ \ 0 \ \ \frac{4}{7} \right)$

"IS NM in Vec(K)?", true

$\mathrm{NM} \left(2 \ \ \frac{12}{7} \ \ 0 \ \ 0 \ \ 0 \ \ \frac{18}{7} \ \ 0 \ \ \frac{24}{7} \right)$

"IS MN in Vec(K)?", false

$\mathrm{MN} \left(2 \ \ 2 \ \ 0 \ \ 0 \ \ 0 \ \ 3 \ \ 0 \ \ 3 \right)$

$$\boldsymbol{\tau} = 13/1 \text{ , rank } = 2 \text{ , ratio } = 13/2 \text{ , } n^2 / r = 25/2$$

$$\boldsymbol{\tau}' = 12/1 \text{ , } r' = 1/2 \text{ , } \boldsymbol{\tau} / n^2 = 13/25$$

$$p^2 = 187/882 \text{ , min } \boldsymbol{\tau} = 4675/882 \text{ , } \boldsymbol{\tau}\text{-check is positive? } 6791/882$$

$$\text{max r} = 882/187 \text{ , } r\text{-check is positive? } 254/441$$

IS N0M0 a combination of T and Omega? , true

$$N_0 M_0 = \frac{1}{2} T + 12\Omega$$

There are, 1, partitions and, 6, ranges, with a group size of, 2

KERNEL HAS LINEAR DIMENSION 8

out of total no. of elements equal to 12

dim span idems 4 vs no. of idems 6

"PT1" = {{1, 3}, {2, 4, 5}}

"RG1" = {3, 5}

"RG2" = {3, 4}

"RG3" = {2, 3}

"RG4" = {1, 5}

"RG5" = {1, 4}

"RG6" = {1, 2}

$$M_C = \begin{pmatrix} \frac{82}{49} & \frac{6}{49} & \frac{-75}{49} & \frac{-13}{21} & \frac{52}{147} \\ \frac{6}{49} & \frac{66}{49} & \frac{-19}{98} & \frac{-25}{42} & \frac{-100}{147} \\ \frac{-75}{49} & \frac{-19}{98} & \frac{321}{196} & \frac{15}{28} & \frac{-22}{49} \\ \frac{-13}{21} & \frac{-25}{42} & \frac{15}{28} & \frac{53}{36} & \frac{-50}{63} \\ \frac{52}{147} & \frac{-100}{147} & \frac{-22}{49} & \frac{-50}{63} & \frac{692}{441} \end{pmatrix} \quad N_C = \begin{pmatrix} \frac{695}{882} & \frac{-187}{882} & \frac{695}{882} & \frac{-187}{882} & \frac{-187}{882} \\ \frac{-187}{882} & \frac{695}{882} & \frac{-187}{882} & \frac{695}{882} & \frac{695}{882} \\ \frac{695}{882} & \frac{-187}{882} & \frac{695}{882} & \frac{-187}{882} & \frac{-187}{882} \\ \frac{-187}{882} & \frac{695}{882} & \frac{-187}{882} & \frac{695}{882} & \frac{695}{882} \\ \frac{-187}{882} & \frac{695}{882} & \frac{-187}{882} & \frac{695}{882} & \frac{695}{882} \end{pmatrix}$$

$$M_C\text{-scaled} = \begin{pmatrix} 1 & \frac{3}{41} & \frac{-75}{82} & \frac{-91}{246} & \frac{26}{123} \\ \frac{1}{11} & 1 & \frac{-19}{132} & \frac{-175}{396} & \frac{-50}{99} \\ \frac{-100}{107} & \frac{-38}{321} & 1 & \frac{35}{107} & \frac{-88}{321} \\ \frac{-156}{371} & \frac{-150}{371} & \frac{135}{371} & 1 & \frac{-200}{371} \\ \frac{39}{173} & \frac{-75}{173} & \frac{-99}{346} & \frac{-175}{346} & 1 \end{pmatrix} \quad N_C\text{-scaled} = \begin{pmatrix} 1 & \frac{-187}{695} & 1 & \frac{-187}{695} & \frac{-187}{695} \\ \frac{-187}{695} & 1 & \frac{-187}{695} & 1 & 1 \\ 1 & \frac{-187}{695} & 1 & \frac{-187}{695} & \frac{-187}{695} \\ \frac{-187}{695} & 1 & \frac{-187}{695} & 1 & 1 \\ \frac{-187}{695} & 1 & \frac{-187}{695} & 1 & 1 \end{pmatrix}$$

$$N_C M_C = \begin{pmatrix} \frac{1}{7} & \frac{-1}{14} & \frac{3}{28} & \frac{-1}{12} & \frac{-2}{21} \\ \frac{-1}{7} & \frac{1}{14} & \frac{-3}{28} & \frac{1}{12} & \frac{2}{21} \\ \frac{1}{7} & \frac{-1}{14} & \frac{3}{28} & \frac{-1}{12} & \frac{-2}{21} \\ \frac{-1}{7} & \frac{1}{14} & \frac{-3}{28} & \frac{1}{12} & \frac{2}{21} \\ \frac{-1}{7} & \frac{1}{14} & \frac{-3}{28} & \frac{1}{12} & \frac{2}{21} \end{pmatrix} \quad M_C N_C = \begin{pmatrix} \frac{1}{7} & \frac{-1}{7} & \frac{1}{7} & \frac{-1}{7} & \frac{-1}{7} \\ \frac{-1}{14} & \frac{1}{14} & \frac{-1}{14} & \frac{1}{14} & \frac{1}{14} \\ \frac{3}{28} & \frac{-3}{28} & \frac{3}{28} & \frac{-3}{28} & \frac{-3}{28} \\ \frac{-1}{12} & \frac{1}{12} & \frac{-1}{12} & \frac{1}{12} & \frac{1}{12} \\ \frac{-2}{21} & \frac{2}{21} & \frac{-2}{21} & \frac{2}{21} & \frac{2}{21} \end{pmatrix} \quad \text{commutator} =$$

$$\begin{pmatrix} 0 & \frac{-1}{14} & \frac{1}{28} & \frac{-5}{84} & \frac{-1}{21} \\ \frac{1}{14} & 0 & \frac{1}{28} & \frac{-1}{84} & \frac{-1}{42} \\ \frac{-1}{28} & \frac{-1}{28} & 0 & \frac{-1}{42} & \frac{-1}{84} \\ \frac{5}{84} & \frac{1}{84} & \frac{1}{42} & 0 & \frac{-1}{84} \\ \frac{1}{21} & \frac{1}{42} & \frac{1}{84} & \frac{1}{84} & 0 \end{pmatrix}$$

Eigenvalues M_C

[0., 0.2079708372, 1.561966029, 2.086724901, 3.842884718]

Eigenvalues N_C

[0., 0., 0., 2.621828106, 1.318081192]

Eigenvalues M_C–scaled

[0., 0.1323256587, 1.025374530, 1.447684396, 2.394615415]

Eigenvalues N_C–scaled

[0., 0., 0., 3.327269625, 1.672730375]

NullSpace M_C

{[1, 1, 1, 1, 1]}

NullSpace N_C

{[−1, 0, 1, 0, 0], [0, −1, 0, 1, 0], [0, −1, 0, 0, 1]}

Eigenvalues M_0

[0.1993457542, 1.553302006, 2.077722402, 3.704131635, 5.465498203]

Eigenvalues N_0

[0., 0., 0., 3., 2.]

NullSpace M_0

{}

NullSpace N_0

{[−1, 0, 1, 0, 0], [0, −1, 0, 1, 0], [0, −1, 0, 0, 1]}

Eigenvalues M

[0., −2.501079729, 2.501079729, −1.004940951, 1.004940951]

Eigenvalues N

[0., 0., 0., 2.449489743, −2.449489743]

NullSpace M

/[-13]\ { [0, ---, 0, 1, 4] } \[2]/

NullSpace N

{[1, 0, -1, 0, 0], [0, 1, 0, -1, 0], [0, 0, 0, -1, 1]}

Harmonic Basis

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$