## T - Run

[2, 3, 2, 3], [4, 4, 1, 1]

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$\tilde{\pi}$ = [1, 1, 1, 1]
**$\delta$** = [2, 2, 2, 2]

POSSIBLE RANKS

1 x 4
2 x 2

BASE DETERMINANT 117/512, .2285156250

*NullSpace* of Δ

{1, 2, 3, 4}

Nullspace of A

[{2, 3},{1, 4}]

STRATIFIED CYCLE COVERS

Degree 0
1

Degree 1
0

Degree 2
v[2] v[3] + v[1] v[4]

Degree 3
v[2] v[4] v[3] + v[1] v[4] v[3] + v[1] v[2] v[4] + v[1] v[2] v[3]

Degree 4
2 v[1] v[2] v[4] v[3]

============================================================

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R: [2, 3, 2, 3]
B: [4, 4, 1, 1]

TRACE TWO = 1

det AT = 0

$$AT = \begin{pmatrix} 0 & \frac{1}{6} & 0 & \frac{5}{6} \\ 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ \frac{5}{6} & \frac{1}{6} & 0 & 0 \\ \frac{5}{6} & 0 & \frac{1}{6} & 0 \end{pmatrix}$$

AT ranking is 1, "vs", 3

Level 2 det = $\frac{1}{512}\,(1+s)\,(3+s)\,(13-6s+s^2)\,(-1+s)\,(-3+s)$

RANK of R is 2

R ranking is 1, "vs", 2

RBAR ranking 1, "vs", 2

RANK of B is 2

B ranking is 1, "vs", 2

BBAR ranking 1, "vs", 2

"R CYCLES", 1 + v[2] v[3]
"B CYCLES", 1 + v[1] v[4]

Eigenvalues

R: [0., 0., 1., −1.]

B: [0., 0., 1., −1.]

NullSpace of R

{[0, 0, 0, 1], [1, 0, 0, 0]}

NullSpace of B

{[0, 0, 1, 0], [0, 1, 0, 0]}

NullSpace of $R^*$

{[1, 0, −1, 0], [0, −1, 0, 1]}

NullSpace of $B^*$

{[1, −1, 0, 0], [0, 0, 1, −1]}

FIXED POINTS DIMENSION 1

$$\text{PROTO-M} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

"RANK of N is ", 3, "RANK of M is ", 4

"RANK of the KERNEL is ", 2

"IdemSolvability Check", 1 "Trace mark", 0, "Rank mark", 2, "for kernel rank", 2

degree 1: $\frac{1}{4}$ ( $v[1] + v[2] + v[3] + v[4]$ )
degree 2: $\frac{1}{2}$ ( $v[2]v[3] + v[1]v[4]$ )

Group spectrum $1 + t + t^2$

## KERNEL STRUCTURE

"PT1" = {{1, 3}, {2, 4}}

"PT2" = {{1, 2}, {3, 4}}

"RG1" = {2, 3}

"RG2" = {1, 4}

$$\pi 2 = [0, 0, 1, 1, 0, 0]$$

supp $\pi 2$ = {3, 4}

$$u2 = [1, 1, 2, 2, 1, 1]$$

supp u2 = {1, 2, 3, 4, 5, 6}

Action of R on ranges, [[1], [1]]
Action of B on ranges, [[2], [2]]

$\boldsymbol{\beta} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

RPARTS [1, 1]

BPARTS [2, 2]

$\boldsymbol{\alpha} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Action of R and B on the blocks of the partitions:

R–BLOCKS,

[2, 3, 2, 3]

B–BLOCKS,

[4, 4, 1, 1]

with invariant measure, [1, 1, 1, 1]

N by blocks, N − check: true

$b_1 = \{1, 2\}$
$b_2 = \{1, 3\}$
$b_3 = \{2, 4\}$
$b_4 = \{3, 4\}$

dim(span of partition vectors), rank($N_0$), rank($N$):  3, 3, 3

$$\text{Centralizer} = \begin{pmatrix} h[1] & 0 & 0 & h[2] \\ 0 & h[1] & h[2] & 0 \\ 0 & h[2] & h[1] & 0 \\ h[2] & 0 & 0 & h[1] \end{pmatrix}$$

## LIE STRUCTURE

Dimension of Lie algebra: 5, Shape: 3 ⊕ 2/1

$$\text{CLB} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

**R and B Cycles. V.**

$$\Omega_R = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \Omega_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$\Omega_R$ in Vec(K)? , {{2, 3}}, true

$\Omega_B$ in Vec(K)? , {{1, 4}}, true

$$V = \begin{pmatrix} \frac{-3}{20} & \frac{9}{20} & \frac{1}{20} & \frac{-7}{20} \\ \frac{-1}{20} & \frac{3}{20} & \frac{7}{20} & \frac{-9}{20} \\ \frac{-9}{20} & \frac{7}{20} & \frac{3}{20} & \frac{-1}{20} \\ \frac{-7}{20} & \frac{1}{20} & \frac{9}{20} & \frac{-3}{20} \end{pmatrix}$$

Omega−checks true, true

$\pi_R = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ vs $\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ $u\Omega_R$ vs $\Omega(I-V)^{-1}$

$\pi_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ vs $\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$ $u\Omega_B$ vs $\Omega(I+V)^{-1}$

**LOCAL GROUPS**

1, "partition", {{1, 3}, {2, 4}}

1, "range", [2, 3], [[3, 2, 3, 2], [2, 3, 2, 3]]

2, "range", [1, 4], [[4, 1, 4, 1], [1, 4, 1, 4]]

2, "partition", {{1, 2}, {3, 4}}

1, "range", [2, 3], [[3, 3, 2, 2], [2, 2, 3, 3]]

2, "range", [1, 4], [[4, 4, 1, 1], [1, 1, 4, 4]]

"group has", 2, "elements" Group element 1,1 = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$g_1 =$ []

$g_2 =$ [[1, 2]]

linear dimension, 2

"Symmetric?", true

Is Z in Vec(K)? true
$\begin{pmatrix} h[1] & h[2] \end{pmatrix}$

"Basis for Z(G)"

1, "coeff", 1

Z[ 1 ] = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2, "coeff", 1

Z[ 2 ] = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Check abelian

1, 2, true

EIGS = $\begin{pmatrix} 1. & 1. \\ 1. & -1. \end{pmatrix}$

PermChars :=

$$\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix}$$

Group spectrum: $1 + t + t^2$

Molien Series to order 10: $1 + t + 2t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 4t^7 + 5t^8 + 5t^9 + 6t^{10}$

n-choose-rank

{1, [1, 2]}, {2, [1, 3]}, {3, [1, 4]}, {4, [2, 3]}, {5, [2, 4]}, {6, [3, 4]}

**KERNEL HIERARCHY**

$\pi 2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$

{3, 4}

$u2 = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}$

{1, 2, 3, 4, 5, 6}

picheck $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

$\pi = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

$\pi 1 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

$u1 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

picheck $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

Column Projections

$P_1 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ idem-checks N0-checks

$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$ idem-checks N0-checks

Row Projections

$$PP_1 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$ idem-checks

$$PP_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$ idem-checks

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad NM = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

CHECKING NULLSPACES

$\pi\boldsymbol{\Delta} = [-1, 1, 1, -1]$

$\ker N_c = \begin{pmatrix} -1 & 1 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$ RB checks

$$\ker M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \begin{pmatrix} -t & s \\ -t & -s \\ t & s \\ t & -s \end{pmatrix}$$ RB checks

$$\ker M_c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} t & s+t & -t \\ s+t & t & -t \\ 0 & s & t \\ s & 0 & t \end{pmatrix}$$ RB checks

$n\pi x^{\dagger} = \begin{pmatrix} 2 & 2 & 0 \end{pmatrix}$

$$RN_0R^* = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \; RR^* = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$B N_0 B^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, BB^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Do these reconstitute N? , true

Ranks: 2, 2, "vs", 2

$$\text{CNM} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Skew T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{Skew Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \quad N_0 = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

"IS NM a combination of T and Omega?", true

$$\text{NM} = 4T + 0\Omega$$

$\Omega \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

$T \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$

"IS NM in Vec(K)?", true

NM $\begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}$

"IS MN in Vec(K)?", true

MN $\begin{pmatrix} 2 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}$

$$\boldsymbol{\tau} = 8/1 \text{ , rank} = 2 \text{ , ratio} = 4/1 \text{ , } n^2 / r = 8/1$$

$$\boldsymbol{\tau}' = 8/1 \text{ , } r' = 1/2 \text{ , } \boldsymbol{\tau} / n^2 = 1/2$$

$$p^2 = 1/4 \text{ , min } \boldsymbol{\tau} = 4/1 \text{ , } \boldsymbol{\tau}\text{-check is positive? } 4/1$$

$$\text{max r} = 4/1 \text{ , } r\text{-check is positive? } 1/2$$

IS N0M0 a combination of T and Omega? , true

$$N_0 M_0 = 0T + 8\Omega$$

There are, 2, partitions and, 2, ranges, with a group size of, 2

KERNEL HAS LINEAR DIMENSION 6
out of total no. of elements equal to 8

dim span idems 4 vs no. of idems 4

"PT1" = {{1, 3}, {2, 4}}

"PT2" = {{1, 2}, {3, 4}}

"RG1" = {2, 3}

"RG2" = {1, 4}

$$M_C = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad N_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{-1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$M_C\text{-scaled} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad N_C\text{-scaled} = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & 1 & \frac{-1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} & 1 & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

$$N_C M_C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_C N_C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{commutator} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues $M_C$

[0., 0., 0., 4.]

Eigenvalues $N_C$

[0., 1., 1., 1.]

Eigenvalues $M_C$–scaled

[0., 0., 0., 4.]

Eigenvalues $N_C$–scaled

[0., 1.333333333, 1.333333333, 1.333333333]

NullSpace $M_C$

{[1, 0, 1, 0], [−1, 0, 0, 1], [1, 1, 0, 0]}

NullSpace $N_C$

{[1, −1, −1, 1]}

Eigenvalues $M_0$

[4., 4., 0., 0.]

Eigenvalues $N_0$

[0., 2., 1., 1.]

NullSpace $M_0$

{[1, 0, 0, −1], [0, 1, −1, 0]}

NullSpace $N_0$

{[−1, 1, 1, −1]}

Eigenvalues M

[2., −2., 2., −2.]

Eigenvalues N

[0., 2., −1., −1.]

NullSpace M

{}

NullSpace N

{[−1, 1, 1, −1]}

Harmonic Basis

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

============================================================

{2, 3}

R: [2, 4, 1, 3]
B: [4, 3, 2, 1]

TRACE TWO = 2

det AT = $\frac{-1}{2}\,(1 + t^2)\,(t)$

$$AT = \begin{pmatrix} 0 & \frac{1}{6} & 0 & \frac{5}{6} \\ 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 \\ \frac{5}{6} & 0 & \frac{1}{6} & 0 \end{pmatrix}$$

AT ranking is 1, "vs", 4

Level 2 det = $\frac{1}{512}\,(13 + 6s + s^2)\,(9 - 2s + s^2)\,(-1 + s)^2$

RANK of R is 4

R ranking is 1, "vs", 4

RBAR ranking 1, "vs", 4

RANK of B is 4

B ranking is 1, "vs", 4

BBAR ranking 1, "vs", 4

"R CYCLES", 1 + v[1] v[2] v[4] v[3]
"B CYCLES", (1 + v[2] v[3]) (1 + v[1] v[4])

Eigenvalues

R: [−1., 1., 1. I, −1. I]

B: [1., −1., 1., −1.]

NullSpace of R

{}

NullSpace of B

{}

NullSpace of $R^*$

{}

NullSpace of $B^*$

{}

FIXED POINTS DIMENSION 2

$$\text{PROTO-M} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad N = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

"RANK of N is ", 4, "RANK of M is ", 4

"RANK of the KERNEL is ", 4

"IdemSolvability Check", 3 "Trace mark", 4, "Rank mark", 4, "for kernel rank", 4

degree 1: $\frac{1}{4}$ ( $v[1] + v[2] + v[3] + v[4]$ )
degree 2: $\frac{1}{4}$ ( $v[1]v[2] + v[1]v[3] + 2v[1]v[4] + 2v[2]v[3] + v[2]v[4] + v[4]v[3]$ )
degree 3 : $\frac{1}{4}$ ( $v[2]v[4]v[3] + v[1]v[4]v[3] + v[1]v[2]v[4] + v[1]v[2]v[3]$ )
degree 4 : 1 ( $v[4]$ ) ( $v[3]$ ) ( $v[1]$ ) ( $v[2]$ )

Group spectrum $1 + t + 2t^2 + t^3 + t^4$

## KERNEL STRUCTURE

"PT1" = {{4}, {1}, {3}, {2}}

"RG1" = {1, 2, 3, 4}

$$\pi 4 = [1]$$

supp $\pi$4 = {1}

$$u4 = [1]$$

supp u4 = {1}

Action of R on ranges, [[1]]
Action of B on ranges, [[1]]

$\boldsymbol{\beta} = (1)$

RPARTS [1]

BPARTS [1]

$\boldsymbol{\alpha} = (1)$

Action of R and B on the blocks of the partitions:

R–BLOCKS,

[4, 3, 1, 2]

B–BLOCKS,

[2, 1, 4, 3]

with invariant measure, [1, 1, 1, 1]

N by blocks, N – check: true

$b_1$ = {4}
$b_2$ = {1}
$b_3$ = {3}
$b_4$ = {2}

dim(span of partition vectors), rank($N_0$), rank($N$): 4, 4, 4

$$\text{Centralizer} = \begin{pmatrix} h[2] & h[3] & h[1] & h[4] \\ h[1] & h[2] & h[4] & h[3] \\ h[3] & h[4] & h[2] & h[1] \\ h[4] & h[1] & h[3] & h[2] \end{pmatrix}$$

## LIE STRUCTURE

Dimension of Lie algebra: 2, Shape: $0 \oplus 2/0$

$$\text{CLB} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

**R and B Cycles. V.**

$$\Omega_R = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \quad \Omega_B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$\Omega_R$ in Vec(K)? , {{1, 2, 3, 4}}, true

$\Omega_B$ in Vec(K)? , {{1, 4}, {2, 3}}, true

$$V = \begin{pmatrix} \frac{-3}{20} & \frac{9}{20} & \frac{1}{20} & \frac{-7}{20} \\ \frac{1}{20} & \frac{-3}{20} & \frac{-7}{20} & \frac{9}{20} \\ \frac{9}{20} & \frac{-7}{20} & \frac{-3}{20} & \frac{1}{20} \\ \frac{-7}{20} & \frac{1}{20} & \frac{9}{20} & \frac{-3}{20} \end{pmatrix}$$

Omega-checks true, true

"I+V is singular"

$\pi_R = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ vs $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ $\quad u\Omega_R$ vs $\Omega(I-V)^{-1}$

$\pi_B = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ vs $\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$ $\quad u\Omega_B$ vs $\Omega(I+V)^{-1}$

## LOCAL GROUPS

1, "partition", {{4}, {1}, {3}, {2}}

1, "range", [1, 2, 3, 4], [[4, 3, 2, 1], [3, 1, 4, 2], [2, 4, 1, 3], [1, 2, 3, 4]]

"group has", 4, "elements"    Group element 1,1 = $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$g_1$ =    [[1, 4], [2, 3]]

$g_2$ =    [[1, 3, 4, 2]]

$g_3$ =    [[1, 2, 4, 3]]

$g_4$ =    []

linear dimension, 4

"Symmetric?", false

Is Z in Vec(K)? true
$\begin{pmatrix} h[4] & h[3] & h[2] & h[1] \end{pmatrix}$

"Basis for Z(G)"

1, "coeff", 1

Z[ 1 ] = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2, "coeff", 1

Z[ 2 ] = $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

3, "coeff", 1

Z[ 3 ] = $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

4, "coeff", 1

Z[ 4 ] = $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Check abelian

1, 2, true
1, 3, true
1, 4, true
2, 3, true
2, 4, true
3, 4, true

$$EIGS = \begin{pmatrix} 1. & 1. & 1. & 1. \\ -1. & 1. & 1.I & -1.I \\ -1. & 1. & 1.I & -1.I \\ 1. & -1. & 1. & -1. \end{pmatrix}$$

PermChars :=

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 2 & 0 & 0 & 6 \\ 0 & 0 & 0 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Group spectrum: $1 + t + 2t^2 + t^3 + t^4$

Molien Series to order 10: $1 + t + 3t^2 + 5t^3 + 10t^4 + 14t^5 + 22t^6 + 30t^7 + 43t^8 + 55t^9 + 73t^{10}$

n-choose-rank

{1, [1, 2, 3, 4]}

**KERNEL HIERARCHY**

$\pi 4 = (1)$

{1}

$u4 = (1)$

{1}

picheck $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

$\pi = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

$\pi 3 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

$u3 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

picheck $\begin{pmatrix} 3 & 3 & 3 & 3 \end{pmatrix}$

$\pi 2 = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix}$

$u2 = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$

picheck $\begin{pmatrix} 6 & 6 & 6 & 6 \end{pmatrix}$

$\pi 1 = \begin{pmatrix} 6 & 6 & 6 & 6 \end{pmatrix}$

$u1 = \begin{pmatrix} \frac{3}{32} & \frac{3}{32} & \frac{3}{32} & \frac{3}{32} \end{pmatrix}$

picheck $\begin{pmatrix} 6 & 6 & 6 & 6 \end{pmatrix}$

Column Projections

$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ idem–checks N0–checks

Row Projections

$PP_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ idem–checks

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad NM = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

CHECKING NULLSPACES

Group case, not scaling MC

$\pi\boldsymbol{\Delta} = [0, 0, 0, 0]$

ker $N_c = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ $\begin{pmatrix} s+t & s+t & s+t & s+t \end{pmatrix}$ RB checks

ker $M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ $\begin{pmatrix} -t & s-t & -t \\ -s & -s & t-s \\ s & t & 0 \\ t & 0 & s \end{pmatrix}$ RB checks

ker $M_c = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} s & t & 0 & 0 \\ 0 & s & t & 0 \\ t & 0 & 0 & s \\ 0 & 0 & s & t \end{pmatrix}$ RB checks

$n\pi x^{\dagger} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$

$$\mathrm{R}N_0\mathrm{R}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathrm{RR}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathrm{B}N_0\mathrm{B}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathrm{BB}^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Do these reconstitute N? , true

Ranks: 4, 4, "vs", 4

$\mathrm{CNM} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Skew T $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Skew Omega $= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$M_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ $N_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

"IS NM a combination of T and Omega?", true

$$\mathrm{NM} = 1T + 8\Omega$$

$\Omega \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

$T \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$

"IS NM in Vec(K)?", true

NM $\begin{pmatrix} 2 & 2 & 2 & 3 \end{pmatrix}$

"IS MN in Vec(K)?", true

MN $\begin{pmatrix} 2 & 2 & 2 & 3 \end{pmatrix}$

$$\boldsymbol{\tau} = 4/1 \text{ , rank} = 4 \text{ , ratio} = 1/1 \text{ , } n^2 / r = 4/1$$

$$\boldsymbol{\tau}' = 12/1 \text{ , } r' = 3/4 \text{ , } \boldsymbol{\tau} / n^2 = 1/4$$

$$p^2 = 1/4 \text{ , min } \boldsymbol{\tau} = 4/1 \text{ , } \boldsymbol{\tau}\text{–check is positive? } 0/1$$

$$\text{max r} = 4/1 \text{ , } r\text{–check is positive? } 0/1$$

IS N0M0 a combination of T and Omega? , true

$$N_0 M_0 = 0T + 4\Omega$$

There are, 1, partitions and, 1, ranges, with a group size of, 4

KERNEL HAS LINEAR DIMENSION 4
out of total no. of elements equal to 4

dim span idems 1 vs no. of idems 1

"PT1" = {{4}, {1}, {3}, {2}}

"RG1" = {1, 2, 3, 4}

$$M_C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad N_C = \begin{pmatrix} \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{pmatrix}$$

$$M_C\text{-scaled} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad N_C\text{-scaled} = \begin{pmatrix} 1 & \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & 1 & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & 1 & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{-1}{3} & 1 \end{pmatrix}$$

$$N_C M_C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad M_C N_C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{commutator} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues $M_C$

[0., 0., 0., 0.]

Eigenvalues $N_C$

[0., 1., 1., 1.]

Eigenvalues $M_C$–scaled

[0., 0., 0., 0.]

Eigenvalues $N_C$–scaled

[0., 1.333333333, 1.333333333, 1.333333333]

NullSpace $M_C$

{[0, 0, 1, 0], [0, 0, 0, 1], [1, 0, 0, 0], [0, 1, 0, 0]}

 

NullSpace $N_c$

{[1, 1, 1, 1]}

Eigenvalues $M_0$

[0., 0., 0., 4.]

Eigenvalues $N_0$

[1., 1., 1., 1.]

NullSpace $M_0$

{[−1, 1, 0, 0], [−1, 0, 1, 0], [−1, 0, 0, 1]}

NullSpace $N_0$

{}

Eigenvalues M

[3., −1., −1., −1.]

Eigenvalues N

[3., −1., −1., −1.]

NullSpace M

{}

NullSpace N

{}

Harmonic Basis

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Commutator(s)

$$1, 2: \quad \text{commutator} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$