

T-Run

[2, 1, 5, 5, 1], [4, 3, 4, 3, 2]

$$\vec{r} = [1, 1, 1, 1, 1]$$

$$\delta = [2, 2, 2, 2, 2]$$

POSSIBLE RANKS

1 x 5

BASE DETERMINANT 725/4096, .1770019531

NullSpace of Δ

{1, 2, 3, 4, 5}

Nullspace of A

$$\det(A) = 1/16$$

STRATIFIED CYCLE COVERS

Degree 0
1

Degree 1
0

Degree 2
 $v[3] v[4] + v[1] v[2]$

Degree 3
 $v[2] v[3] v[5] + v[1] v[4] v[5]$

Degree 4
 $v[2] v[3] v[4] v[5] + v[1] v[2] v[4] v[5] + v[1] v[2] v[3] v[5] + v[1] v[2] v[3] v[4] + v[1] v[3] v[4] v[5]$

Degree 5
 $2 v[1] v[2] v[3] v[4] v[5]$

=====

{2, 3}

R: [2, 3, 4, 5, 1]
B: [4, 1, 5, 3, 2]

TRACE TWO = 1

$$\det AT = \frac{5}{16} (1 + 10t^2 + 5t^4)$$

$$AT = \begin{pmatrix} 0 & \frac{1}{6} & 0 & \frac{5}{6} & 0 \\ \frac{5}{6} & 0 & \frac{1}{6} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 \end{pmatrix}$$

AT ranking is 1, "vs", 5

$$\text{Level 2 det} = \frac{1}{65536} (-100 - 4s - 9s^2 + 4s^3 + s^4) (-1 + s) (116 + 92s + 21s^2 + 7s^3 + 3s^4 + s^5)$$

RANK of R is 5

R ranking is 1, "vs", 5

RBAR ranking 1, "vs", 5

RANK of B is 5

B ranking is 1, "vs", 5

BBAR ranking 1, "vs", 5

"R CYCLES", 1 + v[1] v[2] v[3] v[4] v[5]

"B CYCLES", 1 + v[1] v[2] v[3] v[4] v[5]

Eigenvalues

R: [1., 0.3090169942 + 0.9510565160 I, -0.8090169942 + 0.5877852520 I, -0.8090169942 - 0.5877852520 I, 0.3090169942 - 0.9510565160 I]

B: [1., 0.3090169942 + 0.9510565160 I, -0.8090169942 + 0.5877852520 I, -0.8090169942 - 0.5877852520 I, 0.3090169942 - 0.9510565160 I]

NullSpace of R

{}

NullSpace of B

{}

NullSpace of R*

{}

NullSpace of B*

{}

FIXED POINTS DIMENSION 1

$$\text{PROTO-M} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{N} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

"RANK of N is ", 5, "RANK of M is ", 5

"RANK of the KERNEL is ", 5

"IdemSolvability Check", 3 "Trace mark", 5, "Rank mark", 5, "for kernel rank", 5

degree 1: $\frac{1}{5} (v[1] + v[2] + v[3] + v[4] + v[5])$

degree 2: $\frac{1}{10} (v[1]v[2] + v[1]v[3] + v[1]v[4] + v[1]v[5] + v[2]v[3] + v[2]v[4] + v[2]v[5] + v[3]v[4] + v[3]v[5] + v[4]v[5])$

degree 3: $\frac{1}{10} (v[1]v[2]v[3] + v[1]v[2]v[4] + v[1]v[2]v[5] + v[1]v[3]v[4] + v[1]v[3]v[5] + v[1]v[4]v[5] + v[2]v[3]v[4] + v[2]v[3]v[5] + v[2]v[4]v[5] + v[3]v[4]v[5])$

degree 4: $\frac{1}{5} (v[2]v[3]v[4]v[5] + v[1]v[2]v[4]v[5] + v[1]v[2]v[3]v[5] + v[1]v[2]v[3]v[4] + v[1]v[3]v[4]v[5])$

degree 5: $1 (v[5]) (v[3]) (v[4]) (v[1]) (v[2])$

Group spectrum $1 + t + t^2 + t^3 + t^4 + t^5$

KERNEL STRUCTURE

"PT1" = {{4}, {1}, {5}, {3}, {2}}

"RG1" = {1, 2, 3, 4, 5}

$$\pi_5 = [1]$$

supp $\pi_5 = \{1\}$

$$u_5 = [1]$$

supp $u_5 = \{1\}$

Action of R on ranges, [[1]]

Action of B on ranges, [[1]]

$$\beta = (1)$$

RPARTS [1]

BPARTS [1]

$$\alpha = (1)$$

Action of R and B on the blocks of the partitions:

R-BLOCKS,

[4, 3, 1, 5, 2]

B-BLOCKS,

[2, 5, 4, 1, 3]

with invariant measure, [1, 1, 1, 1, 1]

N by blocks, N - check: true

$$b_1 = \{4\}$$

$$b_2 = \{1\}$$

$$b_3 = \{5\}$$

$$b_4 = \{3\}$$

$$b_5 = \{2\}$$

dim(span of partition vectors), rank(N_0), rank(N): 5, 5, 5

$$\text{Centralizer} = \begin{pmatrix} h[1] & h[2] & h[2] & h[2] & h[2] \\ h[2] & h[1] & h[2] & h[2] & h[2] \\ h[2] & h[2] & h[1] & h[2] & h[2] \\ h[2] & h[2] & h[2] & h[1] & h[2] \\ h[2] & h[2] & h[2] & h[2] & h[1] \end{pmatrix}$$

LIE STRUCTURE

Dimension of Lie algebra: 16, Shape: 15 \oplus 1/0

$$\text{CLB} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

R and B Cycles. V.

$$\Omega_R = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix} \quad \Omega_B = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

Ω_R in Vec(K)? , {{1, 2, 3, 4, 5}}, true

Ω_B in Vec(K)? , {{1, 2, 3, 4, 5}}, true

$$V = \begin{pmatrix} \frac{2}{15} & \frac{7}{15} & \frac{1}{45} & \frac{-19}{45} & \frac{-1}{5} \\ \frac{-7}{15} & \frac{-2}{15} & \frac{19}{45} & \frac{-1}{45} & \frac{1}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{2}{15} & \frac{7}{15} & \frac{-1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{-7}{15} & \frac{-2}{15} & \frac{1}{5} \\ \frac{1}{3} & \frac{-1}{3} & \frac{-1}{9} & \frac{1}{9} & 0 \end{pmatrix}$$

Omega-checks true, true

$$\pi_R = \left(\frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5}\right) \text{ vs } \left(\frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5}\right) \quad u\Omega_R \text{ vs } \Omega(I-V)^{-1}$$

$$\pi_B = \left(\frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5}\right) \text{ vs } \left(\frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5}\right) \quad u\Omega_B \text{ vs } \Omega(I+V)^{-1}$$

LOCAL GROUPS

1, "partition", {{4}, {1}, {5}, {3}, {2}}

1, "range", [1, 2, 3, 4, 5], [[5, 4, 3, 2, 1], [5, 4, 2, 1, 3], [5, 4, 1, 3, 2], [5, 3, 4, 1, 2], [5, 3, 2, 4, 1], [5, 3, 1, 2, 4], [5, 2, 4, 3, 1], [5, 2, 3, 1, 4], [5, 2, 1, 4, 3], [5, 1, 4, 2, 3], [5, 1, 3, 4, 2], [5, 1, 2, 3, 4], [4, 5, 3, 1, 2], [4, 5, 2, 3, 1], [4, 5, 1, 2, 3], [4, 3, 5, 2, 1], [4, 3, 2, 1, 5], [4, 3, 1, 5, 2], [4, 2, 5, 1, 3], [4, 2, 3, 5, 1], [4, 2, 1, 3, 5], [4, 1, 5, 3, 2], [4, 1, 3, 2, 5], [4, 1, 2, 5, 3], [3, 5, 4, 2, 1], [3, 5, 2, 1, 4], [3, 5, 1, 4, 2], [3, 4, 5, 1, 2], [3, 4, 2, 5, 1], [3, 4, 1, 2, 5], [3, 2, 5, 4, 1], [3, 2, 4, 1, 5], [3, 2, 1, 5, 4], [3, 1, 5, 2, 4], [3, 1, 4, 5, 2], [3, 1, 2, 4, 5], [2, 5, 4, 1, 3], [2, 5, 3, 4, 1], [2, 5, 1, 3, 4], [2, 4, 5, 3, 1], [2, 4, 3, 1, 5], [2, 4, 1, 5, 3], [2, 3, 5, 1, 4], [2, 3, 4, 5, 1], [2, 3, 1, 4, 5], [2, 1, 5, 4, 3], [2, 1, 4, 3, 5], [2, 1, 3, 5, 4], [1, 5, 4, 3, 2], [1, 5, 3, 2, 4], [1, 5, 2, 4, 3], [1, 4, 5, 2, 3], [1, 4, 3, 5, 2], [1, 4, 2, 3, 5], [1, 3, 5, 4, 2], [1, 3, 4, 2, 5], [1, 3, 2, 5, 4], [1, 2, 5, 3, 4], [1, 2, 4, 5, 3], [1, 2, 3, 4, 5]]

"group has", 60, "elements" Group element 1,1 = $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

$g_1 =$ [[1, 5], [2, 4]]

$g_2 =$ [[1, 5, 3, 2, 4]]

$g_3 =$ [[1, 5, 2, 4, 3]]

$g_4 =$ [[1, 5, 2, 3, 4]]

$g_5 =$ [[1, 5], [2, 3]]

linear dimension, 17

"Symmetric?", true

Is Z in Vec(K)? true

$(12h[1] - 3h[2] \quad -12h[1] \quad 6h[2] \quad -3h[2] \quad 3h[2] \quad -12h[1] - 3h[2] \quad -12h[1] - 3h[2] \quad 12h[1] + 6h[2] \quad 12h[1] - 3h[2] \quad 3h[2] \quad -1)$

"Basis for Z(G)"

1, "coeff", 12

$$Z[1] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2, "coeff", 3

$$Z[2] = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Check abelian

1, 2, true

$$\text{EIGS} = \begin{pmatrix} 1. & 1. & 1. & 1. & 1. \\ 4. & -1. & -1. & -1. & -1. \end{pmatrix}$$

Group spectrum: $1 + t + t^2 + t^3 + t^4 + t^5$

Molien Series to order 10: $1 + t + 2t^2 + 3t^3 + 5t^4 + 7t^5 + 10t^6 + 13t^7 + 18t^8 + 23t^9 + 31t^{10}$

n-choose-rank

{1, [1, 2, 3, 4, 5]}

KERNEL HIERARCHY

$$\pi_5 = (1)$$

{1}

$$u_5 = (1)$$

{1}

picheck (1 1 1 1 1)

$$\pi = \left(\frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5}\right)$$

$$\pi_4 = (1 \quad 1 \quad 1 \quad 1 \quad 1)$$

$$u_4 = \left(\frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5} \quad \frac{1}{5}\right)$$

picheck (4 4 4 4 4)

$$\pi_3 = (2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2)$$

$$u_3 = \left(\frac{2}{25} \frac{2}{25} \frac{2}{25} \frac{2}{25} \frac{2}{25} \frac{2}{25} \frac{2}{25} \frac{2}{25} \frac{2}{25} \frac{2}{25} \right)$$

picheck (12 12 12 12 12)

$$\pi_2 = (6 \ 6 \ 6 \ 6 \ 6 \ 6 \ 6 \ 6 \ 6 \ 6)$$

$$u_2 = \left(\frac{6}{125} \frac{6}{125} \frac{6}{125} \frac{6}{125} \frac{6}{125} \frac{6}{125} \frac{6}{125} \frac{6}{125} \frac{6}{125} \frac{6}{125} \right)$$

picheck (24 24 24 24 24)

$$\pi_1 = (24 \ 24 \ 24 \ 24 \ 24)$$

$$u_1 = \left(\frac{24}{625} \frac{24}{625} \frac{24}{625} \frac{24}{625} \frac{24}{625} \right)$$

picheck (24 24 24 24 24)

Column Projections

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks} \quad \text{NO-checks}$$

Row Projections

$$PP_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{idem-checks}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad NM = \begin{pmatrix} 4 & 3 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 3 & 3 & 3 & 4 & 3 \\ 3 & 3 & 3 & 3 & 4 \end{pmatrix}$$

CHECKING NULLSPACES

Group case, not scaling MC

$$\pi\Delta = [0, 0, 0, 0, 0]$$

$$\ker N_C = (1 \ 1 \ 1 \ 1 \ 1) \quad (t+s \ t+s \ t+s \ t+s \ t+s) \quad \text{RB checks}$$

$$\ker M_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s-t & -t & -t & -t \\ 0 & s & t & 0 \\ -s & -s & -s & -s+t \\ 0 & t & 0 & s \\ t & 0 & s & 0 \end{pmatrix} \quad \text{RB checks}$$

$$\ker M_C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & t & 0 & s \\ s & t & 0 & 0 & 0 \\ 0 & 0 & s & t & 0 \\ t & 0 & 0 & s & 0 \\ 0 & s & 0 & 0 & t \end{pmatrix} \text{ RB checks}$$

$$n\pi x^\dagger = (1 \ 1 \ 1 \ 1 \ 1)$$

$$RN_0R^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, RR^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$BN_0B^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, BB^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Do these reconstitute N? , true

Ranks: 5, 5, "vs", 5

$$CNM = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ Skew T} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ Skew Omega} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} N_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

"IS NM a combination of T and Omega?", true

$$NM = 1T + 15\Omega$$

$$\Omega \left(\frac{2}{5} \ \frac{1}{2} \ \frac{-7}{5} \ 1 \ 1 \ \frac{3}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{3}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \ \frac{1}{5} \right)$$

$$T \left(1 \ \frac{1}{2} \ -5 \ 2 \ 2 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \right)$$

"IS NM in Vec(K)?", true

$$NM (7 \ 8 \ -26 \ 17 \ 17 \ 10 \ 3 \ 3 \ 10 \ 3 \ 4 \ 3 \ 3 \ 3 \ 3 \ 4)$$

"IS MN in Vec(K)?", true

$$MN (7 \ 8 \ -26 \ 17 \ 17 \ 10 \ 3 \ 3 \ 10 \ 3 \ 4 \ 3 \ 3 \ 3 \ 3 \ 4)$$

$$\tau = 5/1, \text{ rank} = 5, \text{ ratio} = 1/1, n^2 / r = 5/1$$

$$\tau' = 20/1, r' = 4/5, \tau / n^2 = 1/5$$

$$\rho^2 = 1/5, \min \tau = 5/1, \tau\text{-check is positive? } 0/1$$

$$\max r = 5/1, r\text{-check is positive? } 0/1$$

IS N0M0 a combination of T and Omega? , true

$$N_0 M_0 = 0T + 5\Omega$$

There are, 1, partitions and, 1, ranges, with a group size of, 60

KERNEL HAS LINEAR DIMENSION 17
out of total no. of elements equal to 60

dim span idems 1 vs no. of idems 1

"PT1" = {{4}, {1}, {5}, {3}, {2}}

"RG1" = {1, 2, 3, 4, 5}

$$M_C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N_C = \begin{pmatrix} \frac{4}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{4}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{4}{5} & \frac{-1}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{4}{5} & \frac{-1}{5} \\ \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{-1}{5} & \frac{4}{5} \end{pmatrix}$$

$$M_C\text{-scaled} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad N_C\text{-scaled} = \begin{pmatrix} 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & 1 & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & 1 & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & 1 \end{pmatrix}$$

$$N_C M_C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad M_C N_C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{commutator} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalues M_C

[0., 0., 0., 0., 0.]

Eigenvalues N_C

[0., 1., 1., 1., 1.]

Eigenvalues $M_C\text{-scaled}$

[0., 0., 0., 0., 0.]

Eigenvalues $N_C\text{-scaled}$

[0., 1.250000000, 1.250000000, 1.250000000, 1.250000000]

NullSpace M_C

{[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 0, 0, 1]}

NullSpace N_C

{[1, 1, 1, 1, 1]}

Eigenvalues M_0

[0., 0., 0., 0., 5.]

Eigenvalues N_0

[1., 1., 1., 1., 1.]

NullSpace M_0

{[-1, 0, 0, 0, 1], [-1, 0, 0, 1, 0], [-1, 0, 1, 0, 0], [-1, 1, 0, 0, 0]}

NullSpace N_0

{}

Eigenvalues M

[4., -1., -1., -1., -1.]

Eigenvalues N

[4., -1., -1., -1., -1.]

NullSpace M

{}

NullSpace N

{}

Harmonic Basis

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$