

# New Graph

[4, 5, 1, 6, 3, 1], [2, 4, 5, 3, 1, 4]

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$$\pi = [6, 3, 5, 6, 4, 3]$$

POSSIBLE RANKS

$$\begin{matrix} 1 \times 27 \\ 3 \times 9 \end{matrix}$$

BASE DETERMINANT 161757/1048576, .1542634964

*NullSpace* of  $\Delta$

{1, 2, 3, 4, 5, 6}

*Range* of  $\Delta$ :  $[-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$

1. Coloring, {}

**R**: [4, 5, 1, 6, 3, 1]   **B**: [2, 4, 5, 3, 1, 4]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$[-54(\tau^3 + \tau^2), 27(\tau^3 + \tau^2)(\tau - 1), 27(5 + 2\tau + \tau^2)(\tau - 1), -54(\tau^3 + \tau^2), 108(\tau - 1), -27(1 + \tau)(\tau^3 + \tau^2)]$

For  $\tau=1/2$ , [-52, -13, -25, -52, -16, -39] . FixedPtCheck, [52, 13, 25, 52, 16, 39]

$$\det(A + \tau \Delta) = 1(\tau + 1)^3(\tau - 1)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	5 vs 5	5 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 3/4, 0, 0, 1/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [3728/35919, -119305/107757, -11392/11973, 49552/323271, -590144/323271, \\ & 1184512/323271], [-170/2763, 12773/8289, -4532/8289, 6592/24867, 71488/24867, -100352/24867], \\ & [-7504/35919, 85679/107757, -2968/11973, 566224/323271, -410432/323271, -253184/323271], \\ & [2003/35919, -124894/107757, 17060/35919, -525008/323271, -813056/323271, 1553152/323271], \\ & [-913/35919, -13654/107757, 41120/35919, -191936/323271, 825088/323271, -942080/323271], \\ & [1265/2763, 15212/8289, 356/921, 13648/24867, 112384/24867, -191744/24867] ] \$ \times \$ [ [5, 9/2, 11/2, \\ & 6, 9/2, 3/2], [41/8, 15/4, 45/8, 23/4, 21/4, 3/2], [183/32, 123/32, 45/8, 167/32, 165/32, 23/16], [721/128, \\ & 549/128, 333/64, 345/64, 663/128, 167/128], [1411/256, 2163/512, 2733/512, 2869/512, 2547/512, \\ & 345/256], [1383/256, 4233/1024, 5577/1024, 11381/2048, 5181/1024, 2869/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 4, 6}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, y_4, y_5]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, 0, 28/81, 1/81, -26/81], [1/3, -4/9, 28/81, -26/81, 10/81], [0, 0, 1/81, -26/81, 28/81], \\ & [0, 0, -26/81, 28/81, 1/81], [0, 1/3, -26/81, 28/81, -26/81], [0, 0, 1/81, -26/81, 28/81] ] \$ \times \$ [ [8, 0, 4, 6, \\ & 3, 6], [10, 0, 3, 8, 0, 6], [9, 0, 0, 10, 0, 8], [8, 0, 0, 9, 0, 10], [10, 0, 0, 8, 0, 9] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 3, 4, 5}}, net cycles: 1 . order: 5

$$[y_1, y_2, y_3, y_4, y_5, 0]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & [0, 0, 0, 0, 0, 1] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [94/297, -122/297, 13/297, 40/297, -14/297], [-14/297, 94/297, -122/297, 13/297, 40/297], \\ & [13/297, 40/297, -14/297, 94/297, -122/297], [40/297, -14/297, 94/297, -122/297, 13/297], [-122/297, \\ & 13/297, 40/297, -14/297, 94/297], [-14/297, 94/297, -122/297, 13/297, 40/297] ] \$ \times \$ [ [4, 6, 6, 6, 5, 0], \\ & [5, 4, 6, 6, 6, 0], [6, 5, 6, 4, 6, 0], [6, 6, 4, 5, 6, 0], [6, 6, 5, 6, 4, 0] ] \$ \end{aligned}$$

Â» SYNC'D 447/8192 , 0.05456542969

2. Coloring, {2}

**R:** [4, 4, 1, 6, 3, 1] **B:** [2, 5, 5, 3, 1, 4]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\left[ -54 \left( 3 + \tau^2 \right)^2, 27 \left( 3 + \tau^2 \right)^2 \left( -1 + \tau \right), -27 \left( -15 + 7\tau - \tau^2 + \tau^3 \right) \left( 1 + \tau \right) \left( -1 + \tau \right), 54 \left( 3 + \tau^2 \right) \left( 1 + \tau \right) \left( -3 + \tau \right), -108 \left( 3 + \tau \right) \left( -1 + \tau \right)^2, 27 \left( 3 + \tau^2 \right) \left( 1 + \tau \right)^2 \left( -3 + \tau \right) \right]$$

For  $\tau=1/2$ , [-676, -169, -279, -780, -112, -585] . FixedPtCheck, [676, 169, 279, 780, 112, 585]

$$\det(A + \tau \Delta) = 1 \left( 1 + \tau \right)^3 \left( -1 + \tau \right)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$$\$ \left[ \left[ 0, 3/4, 0, 1/4, 0, 0 \right], \left[ 0, 0, 0, 1/4, 3/4, 0 \right], \left[ 1/4, 0, 0, 0, 3/4, 0 \right], \left[ 0, 0, 3/4, 0, 0, 1/4 \right], \left[ 3/4, 0, 1/4, 0, 0, 0 \right], \left[ 1/4, 0, 0, 3/4, 0, 0 \right] \right] \times \$ \left[ \left[ 1, 0, 0, 0, 0, 0 \right], \left[ 0, 1, 0, 0, 0, 0 \right], \left[ 0, 0, 1, 0, 0, 0 \right], \left[ 0, 0, 0, 1, 0, 0 \right], \left[ 0, 0, 0, 0, 1, 0 \right], \left[ 0, 0, 0, 0, 0, 1 \right] \right] \$ =$$

$$\$ \left[ \left[ 54527/635157, -417112/1905471, -378536/635157, 1668352/5716413, -4993280/5716413, 7704064/5716413 \right], \left[ -64109/635157, -34180/1905471, 3837304/1905471, 6013696/5716413, 19850752/5716413, -36485120/5716413 \right], \left[ -150457/635157, 1185608/1905471, 306544/635157, 4756288/5716413, -593792/5716413, -8912384/5716413 \right], \left[ 162233/635157, 726770/1905471, -1290608/211719, -16741760/5716413, -41322752/5716413, 89482240/5716413 \right], \left[ -6571/635157, -451186/1905471, 150656/211719, -2633504/5716413, 3549952/5716413, -3359744/5716413 \right], \left[ 101831/635157, -1959568/1905471, 6108088/635157, 19717312/5716413, 69037696/5716413, -138553856/5716413 \right] \right] \times \$ \left[ \left[ 5, 9/2, 11/2, 9/2, 6, 3/2 \right], \left[ 25/4, 15/4, 39/8, 7/2, 15/2, 9/8 \right], \left[ 57/8, 75/16, 9/2, 107/32, 207/32, 7/8 \right], \left[ 793/128, 171/32, 33/8, 231/64, 441/64, 107/128 \right], \left[ 3281/512, 2379/512, 567/128, 899/256, 909/128, 231/256 \right], \left[ 6819/1024, 9843/2048, 4515/1024, 3523/1024, 13941/2048, 899/1024 \right] \right] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 4, 6}}, net cycles: 0 . order: 3

$$\left[ y_1, 0, y_2, y_3, 0, y_4 \right]$$

$$\mathbf{R} = \$ \left[ \left[ 0, 0, 0, 1, 0, 0 \right], \left[ 0, 0, 0, 1, 0, 0 \right], \left[ 1, 0, 0, 0, 0, 0 \right], \left[ 0, 0, 0, 0, 0, 1 \right], \left[ 0, 0, 1, 0, 0, 0 \right], \left[ 1, 0, 0, 0, 0, 0 \right] \right] \times \$ \left[ \left[ 1, 0, 0, 0, 0, 0 \right], \left[ 0, 0, 0, 0, 0, 0 \right], \left[ 0, 0, 1, 0, 0, 0 \right], \left[ 0, 0, 0, 1, 0, 0 \right], \left[ 0, 0, 0, 0, 0, 0 \right], \left[ 0, 0, 0, 0, 0, 1 \right] \right] \$ = \$ \left[ \left[ 0, -26/81, 28/81, 1/81 \right], \left[ 0, -26/81, 28/81, 1/81 \right], \left[ 0, 28/81, 1/81, -26/81 \right], \left[ 0, 1/81, -26/81, 28/81 \right], \left[ 1/4, 1/81, -26/81, 31/324 \right], \left[ 0, 28/81, 1/81, -26/81 \right] \right] \times \$ \left[ \left[ 8, 0, 4, 9, 0, 6 \right], \left[ 10, 0, 0, 8, 0, 9 \right], \left[ 9, 0, 0, 10, 0, 8 \right], \left[ 8, 0, 0, 9, 0, 10 \right] \right] \$$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_4, y_3, y_5, y_2, y_1, 0]$$

$$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, -20/567, 88/567, -47/567], [0, 0, -47/567, -20/567, 88/567], [0, 0, -47/567, -20/567, 88/567], [0, 1/3, -20/567, 88/567, -236/567], [0, 0, 88/567, -47/567, -20/567], [1/3, -2/3, 88/567, -236/567, 358/567] ] \times \$ [ [4, 6, 6, 3, 8, 0], [8, 4, 3, 0, 12, 0], [12, 8, 0, 0, 7, 0], [7, 12, 0, 0, 8, 0], [8, 7, 0, 0, 12, 0] ] \$$$

Â» SYNC'D 435/4096 , 0.1062011719

3 . Coloring, {3}

**R:** [4, 5, 5, 6, 3, 1] **B:** [2, 4, 1, 3, 1, 4]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '54' ( '3 + \tau' ) , -27' ( ' - 1 + \tau' ) ' ( '3 + \tau' ) , 27' ( '5 + 2\tau + \tau^2' ) , 54' ( '3 + \tau' ) , 108' ( '1 + \tau' ) , 27' ( '3 + \tau' ) ' ( '1 + \tau' ) ' ]$$

For τ=1/2, [28, 7, 25, 28, 24, 21] . FixedPtCheck, [28, 7, 25, 28, 24, 21]

$$\det(A + \tau \Delta) = 1' ( ' - 1 + \tau' )'^2 ( '1 + \tau' )'^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 5	4 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 3/4, 0, 0, 1/4], [3/4, 0, 1/4, 0, 0, 0], [1/4, 0, 0, 3/4, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [295/8307, 24226/74763, -24560/74763, -8128/24921, -32512/74763, 57344/74763], \\ & [-227/10224, -8765/92016, 8987/11502, -1051/1917, 908/5751, -1360/5751], [-15775/66456, \\ & -385225/598104, -37313/74763, -5086/24921, 112568/74763, 8672/74763], [5269/132912, \\ & 701707/1196208, 31475/149526, 30941/24921, -75604/74763, -77008/74763], [-7351/66456, \\ & -1134961/598104, -104705/74763, -21934/24921, 45176/74763, 278240/74763], [7651/10224, \\ & 172765/92016, 24773/11502, 443/1917, -3340/5751, -25264/5751] ] \times \$ [ [15/2, 9/2, 11/2, 6, 2, 3/2], \\ & [6, 45/8, 5, 51/8, 5/2, 3/2], [6, 9/2, 173/32, 219/32, 85/32, 51/32], [825/128, 9/2, 371/64, 777/128, \end{aligned}$$

317/128, 219/128] , [849/128, 2475/512, 331/64, 1605/256, 659/256, 777/512] , [12675/2048, 2547/512, 2737/512, 411/64, 5123/2048, 1605/1024] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 5}, {1, 4, 6}}, net cycles: 2 . order: 6

\$ [ [3, 0, 4, 6, 8, 6] , [6, 0, 8, 3, 4, 6] , [6, 0, 4, 6, 8, 3] , [3, 0, 8, 6, 4, 6] , [6, 0, 4, 3, 8, 6] ] \$

[5 y<sub>1</sub> - 4 y<sub>2</sub> + 5 y<sub>3</sub> - 4 y<sub>4</sub>, 0, 4 y<sub>1</sub>, 4 y<sub>2</sub>, 4 y<sub>3</sub>, 4 y<sub>4</sub>]

$$p = s + s^2 - s^4 - s^5$$

Omega Rank for B : cycles: {{1, 2, 3, 4}}, net cycles: 1 . order: 4

[y<sub>1</sub>, y<sub>4</sub>, y<sub>2</sub>, y<sub>3</sub>, 0, 0]

B = \$ [ [0, 1, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [-2/27, 7/27, -2/27, -2/27] , [-2/27, -2/27, 7/27, -2/27] , [7/27, -2/27, -2/27, -2/27] , [-2/27, -2/27, -2/27, 7/27] , [7/27, -2/27, -2/27, -2/27] , [-2/27, -2/27, 7/27, -2/27] ] \$ x \$ [ [9, 6, 6, 6, 0, 0] , [6, 9, 6, 6, 0, 0] , [6, 6, 6, 9, 0, 0] , [6, 6, 9, 6, 0, 0] ] \$

Â» SYNC'D 81/2048 , 0.03955078125

4 . Coloring, {4}

**R:** [4, 5, 1, 3, 3, 1] **B:** [2, 4, 5, 6, 1, 4]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [ '-54' (' 1 + τ ' )'' (' 3 + τ <sup>2</sup> ' )'' (' - 3 + τ ' )' , 27' (' - 1 + τ ' )'' (' 1 + τ ' )'' (' 3 + τ <sup>2</sup> ' )'' (' - 3 + τ ' )' , 27' (' 1 + τ ' )'' (' 15 + 2τ - 2τ <sup>3</sup> + τ <sup>4</sup> ' )' , 54' (' 3 + τ <sup>2</sup> ' )' <sup>2</sup> , -108' (' - 1 + τ ' )'' (' 3 + τ ' )'' (' 1 + τ ' )' , -27' (' - 1 + τ ' )'' (' 3 + τ <sup>2</sup> ' )' <sup>2</sup> ' ]'

For τ=1/2, [780, 195, 759, 676, 336, 169] . FixedPtCheck, [780, 195, 759, 676, 336, 169]

$$\det(A + \tau \Delta) = 1' (' - 1 + \tau ' )' <sup>3</sup> ' (' 1 + \tau ' )' <sup>2</sup>$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-2756/27477, 1817/9159, 148568/247293, -364048/247293, -156736/247293, 39680/27477], \\ & [-12860/247293, -20497/247293, 887060/2225637, 788768/741879, -507712/741879, -1347584/2225637] \\ & , [9952/27477, -3331/9159, -74200/247293, 238160/247293, -18496/247293, -15104/27477], [305/82431, \\ & -16760/82431, 53188/741879, 31888/82431, 40960/82431, -533248/741879], [4349/82431, 34888/82431, \\ & -552920/741879, -1600/27477, 25984/27477, -431104/741879], [-61/639, 86/639, -1436/5751, \\ & -2416/5751, -1024/5751, 4864/5751] ] \$ \times \$ [ [5, 9/2, 5/2, 6, 9/2, 9/2], [41/8, 15/4, 21/8, 8, 3, 9/2], \\ & [129/32, 123/32, 11/4, 239/32, 93/32, 6], [559/128, 387/128, 83/32, 537/64, 387/128, 717/128], \\ & [1105/256, 1677/512, 1461/512, 3871/512, 1383/512, 1611/256], [69/16, 3315/1024, 2627/1024, \\ & 16907/2048, 1515/512, 11613/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 0 . order: 3

$$[y_1, 0, y_3, y_4, y_2, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \$ = \\ & \$ [ [0, -26/81, 28/81, 1/81], [1/3, -26/81, 28/81, -26/81], [0, 28/81, 1/81, -26/81], [0, 1/81, -26/81, 28/81], [0, 1/81, -26/81, 28/81], [0, 28/81, 1/81, -26/81] ] \$ \times \\ & \$ [ [8, 0, 10, 6, 3, 0], [10, 0, 9, 8, 0, 0], [9, 0, 8, 10, 0, 0], [8, 0, 10, 9, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, y_2, 0, y_3, y_4, y_5]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & [0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, 0, 1/5, -10/189, -104/945], [0, 0, 0, 17/189, -10/189], [1/5, -4/25, -14/125, 506/4725, 46/23625], [0, 0, 0, -10/189, 17/189], [0, 1/5, -4/25, -104/945, 506/4725], [0, 0, 0, 17/189, -10/189] ] \$ \times \\ & \$ [ [4, 6, 0, 6, 5, 6], [5, 4, 0, 12, 0, 6], [0, 5, 0, 10, 0, 12], [0, 0, 0, 17, 0, 10], [0, 0, 0, 10, 0, 17] ] \$ \end{aligned}$$

Â» SYNC'D 221/2048 , 0.1079101562

5 . Coloring, {5}

**R:** [4, 5, 1, 6, 1, 1] **B:** [2, 4, 5, 3, 3, 4]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{bmatrix} -54(\tau+1)^2(\tau-3), & 27(\tau-1)(\tau+1)^2(\tau-3), & 27(\tau-1)(\tau+1)^2(\tau-3), & 27(\tau-1)(\tau+1)^2(\tau-3), & 27(\tau-1)(\tau+1)^2(\tau-3) \\ \tau^2, & -54(\tau+1)^2(\tau-3), & -108(\tau-1)(\tau+1)^2(\tau-3), & -27(\tau+1)^2(\tau-3), & -27(\tau+1)^2(\tau-3) \end{bmatrix}$$

For  $\tau=1/2$ , [60, 15, 19, 60, 16, 45] . FixedPtCheck, [60, 15, 19, 60, 16, 45]

$$\det(A + \tau \Delta) = (\tau - 1 + \tau)^3 (\tau + 1)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\begin{bmatrix} 0, 3/4, 0, 1/4, 0, 0 \\ 0, 0, 0, 3/4, 1/4, 0 \\ 1/4, 0, 0, 0, 3/4, 0 \\ 0, 0, 3/4, 0, 0, 1/4 \\ 1/4, 0, 3/4, 0, 0, 0 \\ 1/4, 0, 0, 3/4, 0, 0 \end{bmatrix} \times \begin{bmatrix} 1, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0, 0 \\ 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0 \\ 0, 0, 1, 0 \\ 0, 0, 0, 0, 0, 1 \end{bmatrix} =$$

$$\begin{bmatrix} 464/1755, -2909/5265, 64/5265, -45776/47385, -33344/47385, 93952/47385 \\ -94/945, 1759/2835, -628/945, 18496/25515, 25024/25515, -38912/25515 \\ -496/12285, -2579/36855, 1384/36855, 31504/331695, -173504/331695, 178432/331695 \\ 1097/12285, -11642/36855, 22852/36855, -417488/331695, -284672/331695, 583936/331695 \\ -263/2457, 1286/7371, -160/7371, 29888/66339, 59648/66339, -90112/66339 \\ -61/945, 2836/2835, -1796/2835, 75664/25515, 46336/25515, -128768/25515 \end{bmatrix} \times \begin{bmatrix} 3, 9/2, 15/2, 6, 9/2, 3/2 \\ 27/8, 9/4, 63/8, 21/4, 27/4, 3/2 \\ 129/32, 81/32, 9, 117/32, 207/32, 21/16 \\ 537/128, 387/128, 243/32, 249/64, 945/128, 117/128 \\ 1017/256, 1611/512, 4329/512, 2049/512, 3303/512, 249/256 \\ 4065/1024, 3051/1024, 2007/256, 8361/2048, 7299/1024, 2049/2048 \end{bmatrix} =$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 4, 6}}, net cycles: 0 . order: 3

$$[y_2, 0, 0, y_1, y_4, y_3]$$

$$\begin{bmatrix} 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 1, 0 \\ 1, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 1 \\ 1, 0, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0, 0 \end{bmatrix} \times \begin{bmatrix} 1, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, 0, 1 \end{bmatrix} = \begin{bmatrix} 0, 10/81, 1/81, -8/81 \\ 1/3, -8/81, 10/81, -26/81 \\ 0, 1/81, -8/81, 10/81 \\ 0, -8/81, 10/81, 1/81 \\ 0, 1/81, -8/81, 10/81 \\ 0, 1/81, -8/81, 10/81 \end{bmatrix} \times \begin{bmatrix} 12, 0, 0, 6, 3, 6 \\ 9, 0, 0, 12, 0, 6 \\ 6, 0, 0, 9, 0, 12 \\ 12, 0, 0, 6, 0, 9 \end{bmatrix} =$$

Omega Rank for B : cycles: {{3, 5}}, net cycles: 0 . order: 4

$$[0, y_1, y_2, y_3, y_4, 0]$$

$$\begin{bmatrix} 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 1, 0 \\ 0, 0, 1, 0, 0, 0 \\ 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0 \end{bmatrix} \times \begin{bmatrix} 0, 0, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0, 0 \\ 0, 0, 1, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0 \\ 0, 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, 0, 1 \end{bmatrix} = \begin{bmatrix} 1/6, -1/6, -13/270, 23/270 \\ 0, 1/6, -11/135, -13/270 \\ 0, 0, -11/135, 16/135 \\ 0, 0, \end{bmatrix}$$

16/135, -11/135], [0, 0, 16/135, -11/135], [0, 1/6, -11/135, -13/270] ] \$ x \$ [ [0, 6, 10, 6, 5, 0], [0, 0, 11, 6, 10, 0], [0, 0, 16, 0, 11, 0], [0, 0, 11, 0, 16, 0] ] \$

Â» SYNC'D 39/256 , 0.1523437500

6 . Coloring, {6}

**R:** [4, 5, 1, 6, 3, 4] **B:** [2, 4, 5, 3, 1, 1]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [-54(' 3 + τ ')(' - 1 + τ ')(' 3 + τ <sup>2</sup> '), 27(' 3 + τ ')(' - 1 + τ ')<sup>2</sup> (' 3 + τ <sup>2</sup> '), 27(' - 15 - 4τ - 2τ <sup>2</sup> + 4τ <sup>3</sup> + τ <sup>4</sup> ')(' - 1 + τ '), 54(' 3 + τ <sup>2</sup> ')<sup>2</sup>, 108(' 3 + 2τ + τ <sup>2</sup> ')(' - 1 + τ ')<sup>2</sup>, 27(' 1 + τ ')(' 3 + τ <sup>2</sup> ')<sup>2</sup> ]'

For τ=1/2, [364, 91, 271, 676, 136, 507] . FixedPtCheck, [364, 91, 271, 676, 136, 507]

det(A + τ Δ) = 1(' 1 + τ ')<sup>2</sup> (' - 1 + τ ')(' 1 + 3τ <sup>2</sup> ')

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	5 vs 5	5 vs 5

bi =

\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 3/4, 0, 0, 1/4], [3/4, 0, 1/4, 0, 0, 0], [3/4, 0, 0, 1/4, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ =

\$ [ [61246/1618335, 349387/1618335, 20804/323667, -98960/323667, -8769088/4855005, 8889344/4855005], [-118997/1618335, 1331386/1618335, 332024/323667, 105040/323667, 2347136/4855005, -12360448/4855005], [-275984/1618335, -74963/1618335, -307840/323667, 194800/323667, 4752512/4855005, -1824256/4855005], [113923/1618335, -1144874/1618335, 33476/323667, 323680/323667, -4089664/4855005, 2004992/4855005], [-3788/1618335, -662591/1618335, -78232/323667, -415472/323667, 6210944/4855005, 3373568/4855005], [773128/1618335, 1267981/1618335, 176792/323667, -325184/323667, 7168256/4855005, -10885888/4855005] ] \$ x \$ [ [13/2, 9/2, 11/2, 9/2, 9/2, 3/2], [47/8, 39/8, 9/2, 43/8, 21/4, 9/8], [189/32, 141/32, 171/32, 173/32, 147/32, 43/32], [741/128, 567/128, 333/64, 655/128, 327/64, 173/128], [3147/512, 2223/512, 2619/512, 2615/512, 2565/512, 655/512], [12279/2048, 9441/2048, 5205/1024, 10471/2048, 315/64, 2615/2048] ] \$



Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, 0, y_5, y_2, y_3, y_4]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 14/27, -13/27], [1/3, -4/9, 1/27, -1/27, 4/27], [0, 0, 1/3, -13/27, 5/27], [0, 0, 0, -13/27, 14/27], [0, 1/3, -4/9, 5/27, -1/27], [0, 0, 0, 14/27, -13/27] ] \$ \times \$ [ [5, 0, 4, 9, 3, 6], [4, 0, 3, 11, 0, 9], [3, 0, 0, 13, 0, 11], [0, 0, 0, 14, 0, 13], [0, 0, 0, 13, 0, 14] ] \$$$

Omega Rank for B : cycles: {{1, 2, 3, 4, 5}}, net cycles: 1 . order: 5

$$[y_5, y_4, y_3, y_2, y_1, 0]$$

$$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [199/3537, 550/3537, -44/3537, 145/3537, -719/3537], [-719/3537, 199/3537, 550/3537, -44/3537, 145/3537], [-44/3537, 145/3537, -719/3537, 199/3537, 550/3537], [145/3537, -719/3537, 199/3537, 550/3537, -44/3537], [550/3537, -44/3537, 145/3537, -719/3537, 199/3537], [550/3537, -44/3537, 145/3537, -719/3537, 199/3537] ] \$ \times \$ [ [7, 6, 6, 3, 5, 0], [5, 7, 3, 6, 6, 0], [6, 5, 6, 7, 3, 0], [3, 6, 7, 5, 6, 0], [6, 3, 5, 6, 7, 0] ] \$$$

Â» SYNC'D 45/1024 , 0.04394531250

7. Coloring, {2, 3}

**R:** [4, 4, 5, 6, 3, 1] **B:** [2, 5, 1, 3, 1, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '-54' (' 3 + \tau ')'' (' 3 + \tau ^2 ')', 27' (' 3 + \tau ')'' (' 3 + \tau ^2 ')'' (' -1 + \tau ')', 27' (' 1 + \tau ')'' (' -15 + 7\tau - \tau ^2 + \tau ^3 ')', 54' (' 1 + \tau ')'' (' 3 + \tau ')'' (' -3 + \tau ')', 108' (' -3 - \tau - \tau ^2 + \tau ^3 ')', 27' (' 1 + \tau ')'' ^2 (' 3 + \tau ')'' (' -3 + \tau ')'' ]'$$

For τ=1/2, [-364, -91, -279, -420, -232, -315] . FixedPtCheck, [364, 91, 279, 420, 232, 315]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')'' ^2 (' -1 + \tau ')'' (' 1 + 3\tau ^2 ')'$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 5	3 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 3/4, 0, 0, 1/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [529/2619, 760/7857, -17816/7857, -5200/2619, -2176/7857, 33536/7857], [-677/2619, \\ & -3722/7857, 33052/7857, -448/2619, 16832/7857, -42496/7857], [-1118/2619, 1597/7857, 24160/7857, \\ & 5216/2619, 20864/7857, -58624/7857], [1114/2619, 10309/7857, -63104/7857, 3440/2619, -79360/7857, \\ & 118784/7857], [16/2619, -3101/7857, 2128/7857, 2624/2619, 10496/7857, -17152/7857], [106/2619, \\ & -16943/7857, 85684/7857, -8224/2619, 97472/7857, -141568/7857] ] \$ \times \$ [ [15/2, 9/2, 11/2, 9/2, 7/2, \\ & 3/2], [57/8, 45/8, 17/4, 33/8, 19/4, 9/8], [225/32, 171/32, 137/32, 129/32, 169/32, 33/32], [951/128, \\ & 675/128, 139/32, 495/128, 325/64, 129/128], [3747/512, 2853/512, 2135/512, 2013/512, 2581/512, \\ & 495/512], [14643/2048, 11241/2048, 2155/512, 8085/2048, 5347/1024, 2013/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 4, 6}, {3, 5}}, net cycles: 2 . order: 6

$$\begin{aligned} & \$ [ [3, 0, 4, 9, 5, 6], [6, 0, 5, 3, 4, 9], [9, 0, 4, 6, 5, 3], [3, 0, 5, 9, 4, 6], [6, 0, 4, 3, 5, 9] ] \$ \\ & [2y_1 - y_2 + 2y_3 - y_4, 0, y_1, y_2, y_3, y_4] \end{aligned}$$

$$p = -s - s^2 + s^4 + s^5$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$\begin{aligned} & \$ [ [9, 6, 6, 3, 3, 0], [9, 9, 3, 0, 6, 0], [9, 9, 0, 0, 9, 0], [9, 9, 0, 0, 9, 0], [9, 9, 0, 0, 9, 0] ] \$ \\ & [y_1, y_2, y_1 - y_3, y_1 - y_2, y_3, 0] \end{aligned}$$

$$p = -s^3 + s^5 \quad p = -s^3 + s^4$$

Â» SYNC'D 387/4096 , 0.09448242188

8 . Coloring, {2, 4}

**R:** [4, 4, 1, 3, 3, 1]   **B:** [2, 5, 5, 6, 1, 4]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$\begin{aligned} & [ [ -54(3 + \tau^2), 27(-1 + \tau)(3 + \tau^2), -27(1 + \tau)(5 - 2\tau + \tau^2), -54(3 + \tau^2), \\ & 108(-1 + \tau), 27(-1 + \tau)(3 + \tau^2) ] \end{aligned}$$

For  $\tau=1/2$ ,  $[-52, -13, -51, -52, -16, -13]$  . FixedPtCheck,  $[52, 13, 51, 52, 16, 13]$

$$\det(A + \tau \Delta) = 1^4 (1 + \tau)^2 (-1 + \tau)^3$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	3 vs 3	4 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-1429/11389, 764/14643, 95512/102501, 11968/922509, -919808/922509, 150016/922509], \\ & [-47/307503, 860/43929, 756776/2767527, 140992/102501, -62464/307503, -3952640/2767527], \\ & [4731/11389, 5732/14643, -6896/34167, 60352/922509, -218240/922509, -366080/922509], \\ & [-1711/102501, -3434/4881, -557216/922509, -77696/922509, 1060096/922509, 91136/307503], \\ & [5717/102501, 1062/1627, 189232/922509, -488672/922509, 20992/922509, -37888/102501], \\ & [-5087/34167, -3508/14643, -266344/307503, -586496/922509, 242560/922509, 1535488/922509] ] \$ \times \\ & \$ [ [5, 9/2, 5/2, 9/2, 6, 9/2], [25/4, 15/4, 21/8, 23/4, 21/4, 27/8], [87/16, 75/16, 11/4, 161/32, 153/32, 69/16], \\ & [685/128, 261/64, 157/64, 369/64, 357/64, 483/128], [2939/512, 2055/512, 363/128, 83/16, 627/128, 1107/256], \\ & [5595/1024, 8817/2048, 1291/512, 2909/512, 10521/2048, 249/64] ] \$ \end{aligned}$$

Check x AllOnes:  $[1, 1, 1, 1, 1, 1]$

Omega Rank for R : cycles:  $\{\{1, 3, 4\}\}$ , net cycles: 1 . order: 3

$$[y_1, 0, y_2, y_3, 0, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ \times \\ & \$ [ [1/81, -26/81, 28/81], [1/81, -26/81, 28/81], [-26/81, 28/81, 1/81], [28/81, 1/81, -26/81], \\ & [28/81, 1/81, -26/81], [-26/81, 28/81, 1/81] ] \$ \times \$ [ [8, 0, 10, 9, 0, 0], [10, 0, 9, 8, 0, 0], [9, 0, 8, 10, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles:  $\{\{1, 2, 5\}, \{4, 6\}\}$ , net cycles: 2 . order: 6

$$\$ [ [4, 6, 0, 3, 8, 6], [8, 4, 0, 6, 6, 3], [6, 8, 0, 3, 4, 6], [4, 6, 0, 6, 8, 3], [8, 4, 0, 3, 6, 6] ] \$$$

$$[-y_4 + 2y_1 - y_2 + 2y_3, y_4, 0, y_1, y_2, y_3]$$

$$p = -s - s^2 + s^4 + s^5$$

$\hat{A} \gg \text{SYNC'D } 205/4096, 0.05004882812$

9. Coloring,  $\{2, 5\}$

**R:** [4, 4, 1, 6, 1, 1]    **B:** [2, 5, 5, 3, 3, 4]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$[ '-54' (' 1 + \tau ' )'' (' 3 + \tau ^ 2 ' )'' (' - 3 + \tau ' )' , 27' (' 1 + \tau ' )'' (' - 1 + \tau ' )'' (' 3 + \tau ^ 2 ' )'' (' - 3 + \tau ' )' , -27' (' - 1 + \tau ' )'' (' 15 + 2\tau - 2\tau ^ 3 + \tau ^ 4 ' )' , 54' (' 1 + \tau ' )'' ^ 2 (' - 3 + \tau ' )'' ^ 2 , 108' (' 3 + \tau ' )'' (' - 1 + \tau ' )'' ^ 2 , 27' (' 1 + \tau ' )'' ^ 3 (' - 3 + \tau ' )'' ^ 2 ' ]'$$

For  $\tau=1/2$ , [780, 195, 253, 900, 112, 675] . FixedPtCheck, [780, 195, 253, 900, 112, 675]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ' )'' ^ 2 (' - 1 + \tau ' )'' ^ 3$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	3 vs 3	3 vs 4

bi =

$$\$ [ [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 0, 1/4, 3/4, 0] , [1/4, 0, 0, 0, 3/4, 0] , [0, 0, 3/4, 0, 0, 1/4] , [1/4, 0, 3/4, 0, 0, 0] , [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\$ [ [907/2409, -4712/7227, -2632/21681, -124096/65043, -90880/65043, 243200/65043] , [-4435/7227, 9308/7227, -60776/65043, 358784/65043, 320000/65043, -659456/65043] , [-157/2409, 56/803, 64/2409, 22784/65043, -22912/65043, 512/65043] , [3485/2409, -17990/7227, 30832/21681, -910336/65043, -726784/65043, 1614848/65043] , [-101/803, 314/2409, 2912/21681, 43808/65043, 61184/65043, -111616/65043] , [-5837/2409, 34000/7227, -40984/21681, 1613696/65043, 1271936/65043, -2908672/65043] ] \$ \times \$ [ [3, 9/2, 15/2, 9/2, 6, 3/2] , [15/4, 9/4, 63/8, 3, 9, 9/8] , [9/2, 45/16, 9, 75/32, 243/32, 3/4] , [555/128, 27/8, 477/64, 153/64, 567/64, 75/128] , [2163/512, 1665/512, 135/16, 303/128, 2079/256, 153/256] , [549/128, 6489/2048, 8055/1024, 2373/1024, 17955/2048, 303/512] ] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 4, 6}}, net cycles: 1 . order: 3

$$[y_1, 0, 0, y_2, 0, y_3]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0] , [0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1] , [1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1/81, 10/81, -8/81] , [1/81, 10/81, -8/81] , [10/81, -8/81, 1/81] , [-8/81, 1/81, 10/81] , [10/81, -8/81, 1/81] , [10/81, -8/81, 1/81] ] \$ \times \$ [ [12, 0, 0, 9, 0, 6] , [6, 0, 0, 12, 0, 9] , [9, 0, 0, 6, 0, 12] ]$$

\$

Omega Rank for B : cycles: {{3, 5}}, net cycles: -1 . order: 2

$$\$ [ [0, 6, 10, 3, 8, 0], [0, 0, 11, 0, 16, 0], [0, 0, 16, 0, 11, 0], [0, 0, 11, 0, 16, 0] ] \$$$

$$[0, 2 y_2, y_1, y_2, y_3, 0]$$

$$p = -s^2 + s^4$$

Â» SYNC'D 27/128 , 0.2109375000

10 . Coloring, {2, 6}

**R:** [4, 4, 1, 6, 3, 4] **B:** [2, 5, 5, 3, 1, 1]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '54' ( '3 + \tau' ) ' (-1 + \tau' ) ' ( '3 + \tau^2' ) , -27' ( '3 + \tau' ) ' (-1 + \tau' )^2 ( '3 + \tau^2' ) , 27' \\ & ( ' -1 + \tau' ) ' ( '1 + \tau' ) ' ( '15 - 9\tau + \tau^2 + \tau^3' ) , 54' ( '1 + \tau' ) ' ( '3 + \tau^2' ) ' ( ' -3 + \tau' ) , 108' ( ' \\ & -3 + \tau^2' ) ' ( ' -1 + \tau' )^2 , 27' ( '1 + \tau' )^2 ( '3 + \tau^2' ) ' ( ' -3 + \tau' ) ' ] \end{aligned}$$

For τ=1/2, [-364, -91, -261, -780, -88, -585] . FixedPtCheck, [364, 91, 261, 780, 88, 585]

$$\det(A + \tau \Delta) = 1' ( ' -1 + \tau' )^2 ( '1 + \tau' )^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 3/4, 0, 0, 1/4], [3/4, 0, 1/4, 0, 0, 0], [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\begin{aligned} & \$ [ [39842/1990293, -24539/1990293, 5125544/5970879, 192016/192609, -20085824/17912637, \\ & -12622592/17912637], [-257321/3980586, 1256639/3980586, 2789684/5970879, 174472/192609, \\ & 17649376/17912637, -46077824/17912637], [-136072/1990293, -184049/1990293, -12021400/5970879, \\ & -167408/192609, 21184/17912637, 55156480/17912637], [870577/3980586, -3944251/3980586, \\ & -12016396/5970879, -141992/192609, 5859232/17912637, 57890176/17912637], [-85099/1990293, \\ & 185842/1990293, 2438936/5970879, -173312/192609, 4322176/17912637, 4235776/17912637], \\ & [364165/3980586, 6847937/3980586, 27775772/5970879, 235576/192609, 5005600/17912637, \\ & -142032512/17912637] ] \$ \times \$ [ [13/2, 9/2, 11/2, 3, 6, 3/2], [7, 39/8, 15/4, 25/8, 15/2, 3/4], [57/8, 21/4, \\ & 135/32, 101/32, 207/32, 25/32], [831/128, 171/32, 255/64, 421/128, 909/128, 101/128], [885/128, \\ & 2493/512, 543/128, 101/32, 1791/256, 421/512], [14181/2048, 2655/512, 4215/1024, 3227/1024, \end{aligned}$$

13995/2048, 101/128] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, 0, y_2, y_3, 0, y_4]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 16/135, -11/135], [0, 0, 16/135, -11/135], [0, 1/4, -11/135, -71/540], [0, 0, -11/135, 16/135], [1/4, -5/16, -71/540, 499/2160], [0, 0, 16/135, -11/135] ] \$ \times \$ [ [5, 0, 4, 12, 0, 6], [4, 0, 0, 11, 0, 12], [0, 0, 0, 16, 0, 11], [0, 0, 0, 11, 0, 16] ] \$$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, 0, y_4, 0]$$

$$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, -47/567, -20/567, 88/567], [0, 88/567, -47/567, -20/567], [0, 88/567, -47/567, -20/567], [1/6, -47/567, -20/567, -13/1134], [0, -20/567, 88/567, -47/567], [0, -20/567, 88/567, -47/567] ] \$ \times \$ [ [7, 6, 6, 0, 8, 0], [8, 7, 0, 0, 12, 0], [12, 8, 0, 0, 7, 0], [7, 12, 0, 0, 8, 0] ] \$$$

Â» SYNC'D 31/256 , 0.1210937500

11 . Coloring, {3, 4}

**R:** [4, 5, 5, 3, 3, 1] **B:** [2, 4, 1, 6, 1, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '54' ('1 + \tau')' ('3 + \tau')' ('-1 + \tau')' ('-3 + \tau')', -27' ('1 + \tau')' ('3 + \tau')' ('-1 + \tau')' ('-3 + \tau')', 27' ('1 + \tau')' ('15 + 2\tau - 2\tau^3 + \tau^4)', -54' ('3 + \tau')' ('-1 + \tau')' ('3 + \tau^2')', 108' ('1 + \tau')'^2 ('3 - 2\tau + \tau^2)', 27' ('3 + \tau')' ('-1 + \tau')'^2 ('3 + \tau^2')' ]'$$

For τ=1/2, [420, 105, 759, 364, 648, 91] . FixedPtCheck, [420, 105, 759, 364, 648, 91]

$$\det(A + \tau \Delta) = 1' ('1 + \tau')'^2 ('-1 + \tau')'^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-5228/82579, 127933/743211, 15760/82579, -214544/743211, 2202560/2229633, \\ & -2144512/2229633], [19871/495474, 112277/1486422, -21904/82579, 22712/247737, \\ & -3852640/2229633, 4064384/2229633], [36238/82579, -396509/743211, -115936/247737, \\ & -873872/743211, -2923840/2229633, 6882560/2229633], [-3117/165158, -394921/1486422, \\ & 51832/82579, -307208/743211, 3622112/2229633, -3382912/2229633], [-11201/82579, 409954/743211, \\ & 10568/247737, 1403200/743211, 112256/2229633, -5261824/2229633], [-15349/165158, \\ & 394319/1486422, -53568/82579, 560888/743211, -3073312/2229633, 2535296/2229633] ] \$ \times \$ [ [15/2, \\ & 9/2, 5/2, 6, 2, 9/2], [9/2, 45/8, 2, 69/8, 7/4, 9/2], [63/16, 27/8, 83/32, 279/32, 61/32, 207/32], [639/128, \\ & 189/64, 85/32, 1071/128, 191/128, 837/128], [1215/256, 1917/512, 631/256, 1071/128, 359/256, \\ & 3213/512], [9153/2048, 3645/1024, 2501/1024, 4455/512, 3179/2048, 3213/512] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 5}}, net cycles: 0 . order: 4

$$[y_4, 0, y_3, y_2, y_1, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \$ = \\ & \$ [ [0, 1/3, 14/27, -22/27], [0, 0, 14/27, -13/27], [0, 0, 14/27, -13/27], [0, 0, -13/27, 14/27], \\ & [0, 0, -13/27, 14/27], [1/3, -2/3, -22/27, 32/27] ] \$ \times \$ [ [3, 0, 10, 6, 8, 0], [0, 0, 14, 3, 10, 0], [0, 0, 13, 0, 14, 0], [0, 0, 14, 0, 13, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, y_2, 0, y_3, 0, y_4]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, 1/9, -4/27, 2/27], [0, 0, 5/27, -4/27], [1/9, -2/27, 2/27, -2/27], [0, 0, -4/27, 5/27], [1/9, -2/27, 2/27, -2/27], \\ & [0, 0, 5/27, -4/27] ] \$ \times \$ [ [9, 6, 0, 6, 0, 6], [0, 9, 0, 12, 0, 6], [0, 0, 0, 15, 0, 12], [0, 0, 0, 12, 0, 15] ] \$ \end{aligned}$$

Â» SYNC'D 5/32 , 0.1562500000

12 . Coloring, {3, 5}

**R:** [4, 5, 5, 6, 1, 1] **B:** [2, 4, 1, 3, 3, 4]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$[ '54' ( '3 + \tau^2 ' )' , -27' ( ' - 1 + \tau ' )'' ( '3 + \tau^2 ' )' , 27' ( ' - 5 + \tau^2 ' )'' ( ' - 1 + \tau ' )' , 54' ( '3 + \tau^2 ' )' , -108' ( '1 + \tau ' )'' ( ' - 1 + \tau ' )' , 27' ( '1 + \tau ' )'' ( '3 + \tau^2 ' )'' ]'$$

For  $\tau=1/2$ , [52, 13, 19, 52, 24, 39] . FixedPtCheck, [52, 13, 19, 52, 24, 39]

$$\det(A + \tau \Delta) = 1' ( '1 + \tau ' )'^2 ( ' - 1 + \tau ' )'^3$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\$ [ [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 0, 3/4, 1/4, 0] , [3/4, 0, 0, 0, 1/4, 0] , [0, 0, 3/4, 0, 0, 1/4] , [1/4, 0, 3/4, 0, 0, 0] , [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\$ [ [-2701/62751, 306266/564759, 110608/62751, 193600/1694277, -2296064/1694277, -1667072/1694277] , [71/9654, -25733/86886, 3676/14481, 244168/130329, -107168/130329, -127616/130329] , [-8278/62751, -163021/564759, -53608/62751, -2642480/1694277, 511936/1694277, 4353280/1694277] , [10255/125502, 743995/1129518, -6252/20917, 1068616/1694277, 2453344/1694277, -4207232/1694277] , [-8902/62751, -1131781/564759, -124640/62751, -2833424/1694277, 2848192/1694277, 7048960/1694277] , [6349/9654, 90997/86886, 4300/4827, 191032/130329, -274784/130329, -249728/130329] ] \$ \times \$ [ [11/2, 9/2, 15/2, 6, 2, 3/2] , [13/2, 33/8, 6, 47/8, 3, 3/2] , [45/8, 39/8, 213/32, 187/32, 81/32, 47/32] , [767/128, 135/32, 201/32, 789/128, 369/128, 187/128] , [371/64, 2301/512, 1737/256, 737/128, 21/8, 789/512] , [12555/2048, 1113/256, 3219/512, 6119/1024, 5775/2048, 737/512] ] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 4, 6}}, net cycles: 0 . order: 3

$$[y_1, 0, 0, y_2, y_4, y_3]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] , [1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, -35/1539, 154/1539, -62/1539] , [1/8, -62/1539, -35/1539, -307/12312] , [1/8, -62/1539, -35/1539, -307/12312] , [0, -62/1539, -35/1539, 154/1539] , [0, 154/1539, -62/1539, -35/1539] , [0, 154/1539, -62/1539, -35/1539] ] \$ \times \$ [ [7, 0, 0, 6, 8, 6] , [14, 0, 0, 7, 0, 6] , [6, 0, 0, 14, 0, 7] , [7, 0, 0, 6, 0, 14] ] \$$$

Omega Rank for B : cycles: {{1, 2, 3, 4}}, net cycles: 1 . order: 4



$$[y_1, y_2, y_3, y_4, 0, 0]$$

$$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [-134/2295, -161/2295, 406/2295, -26/2295], [-26/2295, -134/2295, -161/2295, 406/2295], [-161/2295, 406/2295, -26/2295, -134/2295], [406/2295, -26/2295, -134/2295, -161/2295], [406/2295, -26/2295, -134/2295, -161/2295], [-26/2295, -134/2295, -161/2295, 406/2295] ] \$ \times \$ [ [5, 6, 10, 6, 0, 0], [10, 5, 6, 6, 0, 0], [6, 10, 6, 5, 0, 0], [6, 6, 5, 10, 0, 0] ] \$$$

Â» SYNC'D 171/512, 0.3339843750

13. Coloring, {3, 6}

**R:** [4, 5, 5, 6, 3, 4]    **B:** [2, 4, 1, 3, 1, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ -54 \tau^4 (\tau - 1 + \tau^2)^2, 27 \tau^2 (\tau - 1 + \tau^2)^2 (\tau^2 + \tau + 1)^2, -27 \tau^2 (\tau - 15 - 4\tau - 2\tau^2 + 4\tau^3 + \tau^4)^2, 54 \tau^2 (\tau^2 + \tau + 1)^2 (\tau^2 + \tau + 1)^2, -108 \tau^2 (\tau + 1)^2 (\tau - 3 + \tau^2)^2, 27 \tau^2 (\tau + 1)^2 (\tau^2 + \tau + 1)^2 ]$$

For τ=1/2, [196, 49, 271, 364, 240, 273] . FixedPtCheck, [196, 49, 271, 364, 240, 273]

$$\det(A + \tau \Delta) = (\tau - 1 + \tau^2)^2 (\tau + 1)^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	2 vs 4	4 vs 4

bi =

$$\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 3/4, 0, 0, 1/4], [3/4, 0, 1/4, 0, 0, 0], [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [4201/105327, 116212/315981, -371048/947943, 299008/947943, -765184/947943, 485888/947943], [-2495/105327, -28436/315981, 922648/947943, -622592/947943, 854528/947943, -1011712/947943], [-26255/105327, -188348/315981, 31648/947943, -486080/947943, 1238144/947943, 52736/947943], [5587/105327, 179374/315981, -424400/947943, 1021312/947943, -1124608/947943, -25600/947943], [-12809/105327, -658958/315981, -1044032/947943, -701216/947943, 377600/947943, 3494912/947943], [78865/105327, 629788/315981, 2007544/947943, -272960/947943, 358016/947943, -4656640/947943] ] \$ \times \$ [ [9, 9/2, 11/2, 9/2, 2, 3/2], [27/4, 27/4, 31/8, 6, 5/2, 9/8], [45/8, 81/16, 41/8, 225/32, 85/32, 3/2], [891/128, 135/32, 95/16, 357/64, 163/64, 225/128], [3933/512, 2673/512, 617/128,$$

171/32, 325/128, 357/256] , [6723/1024, 11799/2048, 2377/512, 6333/1024, 5141/2048, 171/128] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}, {3, 5}}, net cycles: 2 . order: 2

\$ [ [0, 0, 4, 9, 8, 6] , [0, 0, 8, 6, 4, 9] , [0, 0, 4, 9, 8, 6] , [0, 0, 8, 6, 4, 9] ] \$

[0, 0, 4 y<sub>1</sub> , -15 y<sub>1</sub> + 16 y<sub>2</sub> , -16 y<sub>1</sub> + 16 y<sub>2</sub> , 4 y<sub>2</sub>]

$$p' = s - s^3 \quad p = -s + s^3$$

Omega Rank for B : cycles: {{1, 2, 3, 4}}, net cycles: 1 . order: 4

[y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, 0, 0]

B = \$ [ [0, 1, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [-2/27, 4/27, -2/27, 1/27] , [1/27, -2/27, 4/27, -2/27] , [4/27, -2/27, 1/27, -2/27] , [-2/27, 1/27, -2/27, 4/27] , [4/27, -2/27, 1/27, -2/27] , [4/27, -2/27, 1/27, -2/27] ] \$ x \$ [ [12, 6, 6, 3, 0, 0] , [6, 12, 3, 6, 0, 0] , [3, 6, 6, 12, 0, 0] , [6, 3, 12, 6, 0, 0] ] \$

Â» SYNC'D 81/2048 , 0.03955078125

14 . Coloring, {4, 5}

**R:** [4, 5, 1, 3, 1, 1]    **B:** [2, 4, 5, 6, 3, 4]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [ '-54' ( ' 1 + τ ' )'' ( ' - 3 + τ ' )' 2 , 27' ( ' - 1 + τ ' )'' ( ' 1 + τ ' )'' ( ' - 3 + τ ' )' 2 , 27' ( ' - 15 + 4τ - 2τ 2 - 4τ 3 + τ 4 ' )' , 54' ( ' 3 + τ 2 ' )'' ( ' - 3 + τ ' )' , 108' ( ' - 1 + τ ' )'' ( ' 3 + τ ' )' , -27' ( ' - 1 + τ ' )'' ( ' 3 + τ 2 ' )'' ( ' - 3 + τ ' )' ]'

For τ=1/2, [-300, -75, -223, -260, -112, -65] . FixedPtCheck, [300, 75, 223, 260, 112, 65]

$$\det(A + \tau \Delta) = 1' ( ' - 1 + \tau ' )' 4 ( ' 1 + \tau ' )'$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	3 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 1/4, 0, 0, 3/4], [1/4, 0, 3/4, 0, 0, 0], \\ & [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-2/29, -61/87, -1528/783, -3440/2349, 5056/2349, 4864/2349], [-28/261, 5/87, -6428/7047, \\ & -3392/2349, 1472/2349, 12800/7047], [22/29, 91/87, 2888/783, 7216/2349, -11072/2349, -8960/2349], \\ & [-1/3, -2/3, -92/81, -208/81, 128/81, 256/81], [37/87, 40/29, 6344/2349, 11392/2349, -5504/2349, \\ & -16384/2349], [-17/29, -236/261, -2084/783, -4880/2349, 6784/2349, 7936/2349] ] \$ \times \$ [ [3, 9/2, 9/2, 6, \\ & 9/2, 9/2], [27/8, 9/4, 39/8, 15/2, 9/2, 9/2], [111/32, 81/32, 21/4, 189/32, 135/32, 45/8], [483/128, \\ & 333/128, 297/64, 447/64, 585/128, 567/128], [873/256, 1449/512, 2649/512, 3183/512, 2115/512, \\ & 1341/256], [3723/1024, 2619/1024, 1191/256, 14139/2048, 2349/512, 9549/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 0 . order: 3

$$[y_4, 0, y_2, y_3, y_1, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, \\ & 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, \\ & 0, 0, 0] ] \$ = & \$ [ [0, 10/81, 1/81, -8/81], [1/3, -8/81, 10/81, -26/81], [0, 1/81, -8/81, 10/81], [0, -8/81, \\ & 10/81, 1/81], [0, 1/81, -8/81, 10/81], [0, 1/81, -8/81, 10/81] ] \$ \times \$ [ [12, 0, 6, 6, 3, 0], [9, 0, 6, 12, 0, 0], \\ & [6, 0, 12, 9, 0, 0], [12, 0, 9, 6, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}, {3, 5}}, net cycles: 1 . order: 2

$$\begin{aligned} & \$ [ [0, 6, 4, 6, 5, 6], [0, 0, 5, 12, 4, 6], [0, 0, 4, 6, 5, 12], [0, 0, 5, 12, 4, 6], [0, 0, 4, 6, 5, 12] ] \$ \\ & [0, -2 y_1 + 4 y_2 - y_3, y_1, 4 y_1 - 2 y_2, y_2, y_3] \end{aligned}$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

Â» SYNC'D 21/512, 0.04101562500

15 . Coloring, {4, 6}

**R:** [4, 5, 1, 3, 3, 4]    **B:** [2, 4, 5, 6, 1, 1]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$\begin{aligned} & [ [ -54 ( ' 3 + \tau^2 ' ) ' , 27 ( ' 3 + \tau^2 ' ) ' ( ' - 1 + \tau ' ) ' , 27 ( ' 1 + \tau ' ) ' ( ' - 5 + \tau^2 ' ) ' , -54 ( ' 3 + \tau \\ & 2 ' ) ' , 108 ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' , 27 ( ' 3 + \tau^2 ' ) ' ( ' - 1 + \tau ' ) ' ] ' \end{aligned}$$

For  $\tau=1/2$ ,  $[-52, -13, -57, -52, -24, -13]$  . FixedPtCheck,  $[52, 13, 57, 52, 24, 13]$

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')'' (' - 1 + \tau ')'^2 (' 1 + 3\tau^2 ')'$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [ [-190609/4437693, -51556/4437693, 3744724/13313079, -16427744/39939237, \\ & -3447488/39939237, 12299776/39939237], [-120490/4437693, 69593/4437693, 1500088/13313079, \\ & 18062704/39939237, -22488320/39939237, 1862656/39939237], [1441892/4437693, -723721/4437693, \\ & -2393564/13313079, 2149984/39939237, -20789696/39939237, 20836096/39939237], [25148/4437693, \\ & -827077/4437693, 838768/13313079, 5354560/39939237, 20203648/39939237, -19377920/39939237], \\ & [-126376/4437693, 2021099/4437693, -778964/13313079, 4149280/39939237, 14433856/39939237, \\ & -31819520/39939237], [-304009/4437693, 199076/4437693, -5639180/13313079, -5032016/39939237, \\ & 4380352/39939237, 19992832/39939237] ] \$ \times \$ [ [13/2, 9/2, 5/2, 9/2, 9/2, 9/2], [59/8, 39/8, 9/4, 49/8, \\ & 3, 27/8], [171/32, 177/32, 73/32, 203/32, 93/32, 147/32], [793/128, 513/128, 37/16, 849/128, 99/32, \\ & 609/128], [3311/512, 2379/512, 1245/512, 2941/512, 1401/512, 2547/512], [13089/2048, 9933/2048, \\ & 2171/1024, 12995/2048, 3057/1024, 8823/2048] ] \$ \end{aligned}$$

Check x AllOnes:  $[1, 1, 1, 1, 1, 1]$

Omega Rank for R : cycles:  $\{\{1, 3, 4\}\}$ , net cycles: 0 . order: 3

$$[y_4, 0, y_3, y_1, y_2, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, -95/1053, 40/1053, 94/1053], [1/3, -95/1053, 40/1053, -257/1053], [0, 40/1053, 94/1053, -95/1053], \\ & [0, 94/1053, -95/1053, 40/1053], [0, 94/1053, -95/1053, 40/1053], [0, -95/1053, 40/1053, 94/1053] ] \$ \times \$ [ [5, 0, 10, 9, 3, 0], [10, 0, 12, 5, 0, 0], [12, 0, 5, 10, 0, 0], [5, 0, 10, 12, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles:  $\{\{1, 2, 4, 6\}\}$ , net cycles: 0 . order: 4

$$[y_1, y_2, 0, y_3, y_4, y_5]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 1] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, 173/7749, 821/7749, -583/7749, -124/7749], [0, -124/7749, 173/7749, 821/7749, -583/7749], [1/5, -583/7749, -124/7749, 173/7749, -3644/38745], [0, -583/7749, -124/7749, 173/7749, 821/7749], \\ & [0, 821/7749, -583/7749, -124/7749, 173/7749], [0, 821/7749, -583/7749, -124/7749, 173/7749] ] \$ \times \$ [ [7, 6, 0, 3, 5, 6], [11, 7, 0, 6, 0, 3], [3, 11, 0, 7, 0, 6], [6, 3, 0, 11, 0, 7], [7, 6, 0, 3, 0, 11] ] \$ \end{aligned}$$

Â» SYNC'D 99/512 , 0.1933593750

16 . Coloring, {5, 6}

**R:** [4, 5, 1, 6, 1, 4] **B:** [2, 4, 5, 3, 3, 1]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [-54' ( ' 3 + τ ' )'' ( ' - 1 + τ ' )'' ( ' - 3 + τ ' )'' ( ' 1 + τ ' )' , 27' ( ' 3 + τ ' )'' ( ' - 1 + τ ' )' 2 ' ( ' - 3 + τ ' )'' ( ' 1 + τ ' )' , 27' ( ' 15 - 2τ + 2τ 3 + τ 4 ' )'' ( ' - 1 + τ ' )' , 54' ( ' 3 + τ 2 ' )'' ( ' - 3 + τ ' )'' ( ' 1 + τ ' )' , -108' ( ' - 1 + τ ' )' 2 ' ( ' 3 + 2τ + τ 2 ' )' , 27' ( ' 3 + τ 2 ' )'' ( ' - 3 + τ ' )'' ( ' 1 + τ ' )' 2 ' ]'

For τ=1/2, [-420, -105, -229, -780, -136, -585] . FixedPtCheck, [420, 105, 229, 780, 136, 585]

det(A + τ Δ) = 1' ( ' 1 + 3τ 2 ' )'' ( ' - 1 + τ ' )' 2 ' ( ' 1 + τ ' )'

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

\$ [ [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 0, 3/4, 1/4, 0] , [1/4, 0, 0, 0, 3/4, 0] , [0, 0, 3/4, 0, 0, 1/4] , [1/4, 0, 3/4, 0, 0, 0] , [3/4, 0, 0, 1/4, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ =

\$ [ [1849/4485, -1546/2691, -6668/13455, -19472/13455, -23872/40365, 110336/40365] , [-1238/4485, 2423/2691, -1604/13455, 464/13455, 31424/40365, -51712/40365] , [-131/4485, -262/2691, -668/13455, 4768/13455, -12352/40365, 512/3105] , [697/4485, -1366/2691, 10096/13455, -7376/13455, -52096/40365, 59648/40365] , [-677/4485, 440/2691, 1204/13455, 6016/13455, 32576/40365, -4096/3105] , [-1238/4485, 3251/2691, -5744/13455, 37264/13455, 97664/40365, -228352/40365] ] \$ x \$ [ [9/2, 9/2, 15/2, 9/2, 9/2, 3/2] , [33/8, 27/8, 27/4, 39/8, 27/4, 9/8] , [135/32, 99/32, 279/32, 123/32, 189/32, 39/32] , [585/128, 405/128, 117/16, 471/128, 117/16, 123/128] , [2241/512, 1755/512, 4221/512, 1923/512, 3213/512, 471/512] , [8847/2048, 6723/2048, 963/128, 7977/2048, 7209/1024, 1923/2048] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 4

[y<sub>1</sub>, 0, 0, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

R = \$ [ [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1] , [1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, -4/27, 5/27] , [1/3, -1, -13/27, 32/27] , [0, 1/3, 5/27, -13/27] , [0, 0, 5/27, -4/27] , [0,

$$1/3, 5/27, -13/27], [0, 0, -4/27, 5/27] ] \times [ [9, 0, 0, 9, 3, 6], [3, 0, 0, 15, 0, 9], [0, 0, 0, 12, 0, 15], [0, 0, 0, 15, 0, 12] ]$$

Omega Rank for B : cycles: {{3, 5}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, y_5, 0]$$

$$B = [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0] ] \times [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] = [ [0, 1/3, -2/3, -56/135, 106/135], [0, 0, 1/3, 16/135, -56/135], [0, 0, 0, 16/135, -11/135], [0, 0, 0, -11/135, 16/135], [0, 0, 0, -11/135, 16/135], [1/3, -2/3, 1, 106/135, -191/135] ] \times [ [3, 6, 10, 3, 5, 0], [0, 3, 8, 6, 10, 0], [0, 0, 16, 3, 8, 0], [0, 0, 11, 0, 16, 0], [0, 0, 16, 0, 11, 0] ]$$

Â» SYNC'D 9/128 , 0.07031250000

17 . Coloring, {2, 3, 4}

**R:** [4, 4, 5, 3, 3, 1] **B:** [2, 5, 1, 6, 1, 4]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[ '-54(' - 1 + \tau ')^2 (' 3 + \tau ')^2, 27(' - 1 + \tau ')^2 (' 3 + \tau ')^2, 27(' 1 + \tau ')^2 (' 5 - 2\tau + \tau^2 ')^2, -54(' - 1 + \tau ')^2 (' 3 + \tau ')^2, 108(' 1 + \tau^2 ')^2, 27(' - 1 + \tau ')^2 (' 3 + \tau ')^2 ]$$

For τ=1/2, [28, 7, 51, 28, 40, 7] . FixedPtCheck, [28, 7, 51, 28, 40, 7]

$$\det(A + \tau \Delta) = 1^2 (' - 1 + \tau ')^2 (' 1 + 3\tau^2 ')^2 (' 1 + \tau ')^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 5

bi =

$$[ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], [1/4, 0, 0, 3/4, 0, 0] ] \times [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] =$$

$$[ [-32/1065, 6763/22365, 436/7455, -3008/22365, 448/1917, -26368/67095], [53/1065, 1154/4473, 916/2485, -848/7455, -6976/9585, 13568/67095], [379/1065, 2152/22365, -824/7455, -2288/22365, -37888/67095, 4864/13419], [-56/1065, -11861/22365, -1012/7455, -752/3195, 8896/13419, 22016/67095], [1/1065, 4042/22365, 3208/7455, 15856/22365, -13696/67095, -72448/67095], [-31/213, -130/639, -300/497, 352/4473, 1984/13419, 10240/13419] ] \times [ [15/2, 9/2, 5/2, 9/2, 7/2, 9/2], [45/8, 45/8, 2, 51/8, 4, 27/8], [171/32, 135/32, 83/32, 171/32, 151/32, 153/32] ]$$

[855/128, 513/128, 161/64, 765/128, 61/16, 513/128] , [2943/512, 2565/512, 1253/512, 2907/512, 1861/512, 2295/512] , [11637/2048, 8829/2048, 149/64, 12393/2048, 2237/512, 8721/2048] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 5}}, net cycles: 0 . order: 4

$$[y_1, 0, y_2, y_3, y_4, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0] ] \\ & \$ \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0] ] \\ & \$ = \$ [ [0, 1/3, 14/27, -22/27] , [0, 1/3, 14/27, -22/27] , [0, 0, 14/27, -13/27] , [0, 0, -13/27, 14/27] \\ & , [0, 0, -13/27, 14/27] , [1/3, -1, -22/27, 41/27] ] \$ \times \$ [ [3, 0, 10, 9, 5, 0] , [0, 0, 14, 3, 10, 0] , [0, 0, 13, 0, 14, 0] , [0, 0, 14, 0, 13, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}, {1, 2, 5}}, net cycles: 2 . order: 6

$$\$ [ [9, 6, 0, 3, 3, 6] , [3, 9, 0, 6, 6, 3] , [6, 3, 0, 3, 9, 6] , [9, 6, 0, 6, 3, 3] , [3, 9, 0, 3, 6, 6] ] \$$$

$$[y_2, -y_2 + 2y_1 - y_3 + 2y_4, 0, y_1, y_3, y_4]$$

$$p = -s - s^2 + s^4 + s^5$$

Â» SYNC'D 87/1024 , 0.08496093750

18 . Coloring, {2, 3, 5}

**R:** [4, 4, 5, 6, 1, 1] **B:** [2, 5, 1, 3, 3, 4]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & '[ -54(3 + \tau^2)^2, 27(-1 + \tau)(3 + \tau^2)^2, 27(-1 + \tau)(15 + 2\tau - 2\tau^3 + \tau^4) \\ & ', 54(1 + \tau)(3 + \tau^2)(-3 + \tau), -108(-1 + \tau)(-3 - \tau - \tau^2 + \tau^3), 27(1 + \tau)^2(3 + \tau^2)(-3 + \tau) ]' \end{aligned}$$

For τ=1/2, [-676, -169, -253, -780, -232, -585] . FixedPtCheck, [676, 169, 253, 780, 232, 585]

$$\det(A + \tau \Delta) = 1(1 + 3\tau^2)(-1 + \tau)^2(1 + \tau)$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 3/4, 0, 0, 1/4], [1/4, 0, 3/4, 0, 0, 0], \\ & [1/4, 0, 0, 3/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-29180/156123, 3511/468369, 675968/468369, 880640/1405107, -1324672/1405107, \\ & -1279744/1405107], [52078/156123, -422027/468369, -366364/468369, 1778336/1405107, \\ & -1208896/1405107, 1379072/1405107], [-4250/156123, 96229/468369, -107428/468369, \\ & -1482832/1405107, -576448/1405107, 2183168/1405107], [-121241/156123, 950914/468369, \\ & 413276/468369, 1256912/1405107, 5229632/1405107, -9435904/1405107], [27844/156123, \\ & -121229/468369, -449068/468369, -1338544/1405107, 1317440/1405107, 1533440/1405107], \\ & [270763/156123, -1485566/468369, -1034320/468369, -1797328/1405107, -7396864/1405107, \\ & 14369024/1405107] ] \$ \times \$ [ [11/2, 9/2, 15/2, 9/2, 7/2, 3/2], [55/8, 33/8, 6, 29/8, 21/4, 9/8], [195/32, \\ & 165/32, 213/32, 115/32, 147/32, 29/32], [815/128, 585/128, 393/64, 447/128, 177/32, 115/128], \\ & [3181/512, 2445/512, 3465/512, 1745/512, 2541/512, 447/512], [13383/2048, 9543/2048, 6429/1024, \\ & 6967/2048, 675/128, 1745/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 4, 6}}, net cycles: 0 . order: 3

$$[y_1, 0, 0, y_2, y_3, y_4]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, -25/162, 29/162, 1/81], [0, -25/162, 29/162, 1/81], [1/5, 1/81, -25/162, -17/810], [0, \\ & 1/81, -25/162, 29/162], [0, 29/162, 1/81, -25/162], [0, 29/162, 1/81, -25/162] ] \$ \times \$ [ [7, 0, 0, 9, 5, 6], \\ & [11, 0, 0, 7, 0, 9], [9, 0, 0, 11, 0, 7], [7, 0, 0, 9, 0, 11] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 3, 5}}, net cycles: 0 . order: 4

$$[y_1, y_5, y_2, y_4, y_3, 0]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & [0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, -161/2295, 406/2295, -26/2295, -134/2295], [0, -134/2295, -161/2295, 406/2295, \\ & -26/2295], [0, 406/2295, -26/2295, -134/2295, -161/2295], [0, -26/2295, -134/2295, -161/2295, \\ & 406/2295], [0, -26/2295, -134/2295, -161/2295, 406/2295], [1/3, -134/2295, -161/2295, 406/2295, \\ & -791/2295] ] \$ \times \$ [ [5, 6, 10, 3, 3, 0], [10, 5, 6, 0, 6, 0], [6, 10, 6, 0, 5, 0], [6, 6, 5, 0, 10, 0], [5, 6, 10, 0, \\ & 6, 0] ] \$ \end{aligned}$$

Â» SYNC'D 381/2048 , 0.1860351562

19 . Coloring, {2, 3, 6}



**R:** [4, 4, 5, 6, 3, 4] **B:** [2, 5, 1, 3, 1, 1]

' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$[ '-54' (' 3 + \tau ' )' ^ 2 ' (' - 1 + \tau ' )' , 27' (' 3 + \tau ' )' ^ 2 ' (' - 1 + \tau ' )' ^ 2 , 27' (' 15 - 9\tau + \tau ^ 2 + \tau ^ 3 ' )' ' (' 1 + \tau ' )' , -54' (' 3 + \tau ' )' ' (' 1 + \tau ' )' ' (' - 3 + \tau ' )' , 108' (' 3 + \tau ^ 2 ' )' , -27' (' 3 + \tau ' )' ' (' 1 + \tau ' )' ^ 2 ' (' - 3 + \tau ' )' ]'$$

For  $\tau=1/2$ , [196, 49, 261, 420, 208, 315] . FixedPtCheck, [196, 49, 261, 420, 208, 315]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ' )' ^ 3 ' (' - 1 + \tau ' )' ^ 2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	2 vs 4	4 vs 4

bi =

$$\$ [ [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 0, 1/4, 3/4, 0] , [3/4, 0, 0, 0, 1/4, 0] , [0, 0, 3/4, 0, 0, 1/4] , [3/4, 0, 1/4, 0, 0, 0] , [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\$ [ [4/759, -7/207, 2384/6831, -1136/2277, 1088/2277, -1792/6831] , [-49/253, 148/207, 1060/759, 4816/6831, -2560/6831, -15104/6831] , [232/2277, -337/621, -60080/20493, 688/20493, -21184/20493, 90368/20493] , [448/759, -439/207, -31532/6831, -9280/6831, -6848/6831, 19456/2277] , [-389/2277, 284/621, 34312/20493, 10624/20493, 18560/20493, -68608/20493] , [-20/33, 35/9, 2908/297, 224/99, 64/33, -5120/297] ] \$ \times \$ [ [9, 9/2, 11/2, 3, 7/2, 3/2] , [63/8, 27/4, 25/8, 15/4, 19/4, 3/4] , [207/32, 189/32, 4, 123/32, 187/32, 15/16] , [1035/128, 621/128, 139/32, 213/64, 695/128, 123/128] , [2061/256, 3105/512, 1973/512, 1779/512, 2419/512, 213/256] , [7227/1024, 6183/1024, 1939/512, 7653/2048, 1411/256, 1779/2048] ] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}, {3, 5}}, net cycles: 2 . order: 2

$$\$ [ [0, 0, 4, 12, 5, 6] , [0, 0, 5, 6, 4, 12] , [0, 0, 4, 12, 5, 6] , [0, 0, 5, 6, 4, 12] ] \$$$

$$[0, 0, y_1, -2 y_1 + 4 y_2, y_2, 4 y_1 - 2 y_2]$$

$$p = -s + s^3 \quad p' = -s + s^3$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, 0, y_4, 0]$$

$$\mathbf{B} = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] = \$ [ [0, 10/81, 1/81, -8/81], [0, -8/81, 10/81, 1/81], [0, 1/81, -8/81, 10/81], [1/6, -8/81, 10/81, -25/162], [0, 1/81, -8/81, 10/81], [0, 1/81, -8/81, 10/81] ] \times \$ [ [12, 6, 6, 0, 3, 0], [9, 12, 0, 0, 6, 0], [6, 9, 0, 0, 12, 0], [12, 6, 0, 0, 9, 0] ] \$$$

Â» SYNC'D 33/512 , 0.06445312500

20 . Coloring, {2, 4, 5}

**R:** [4, 4, 1, 3, 1, 1]    **B:** [2, 5, 5, 6, 3, 4]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[ '-54' (' 1 + \tau ')'' (' - 3 + \tau ')', 27' (' 1 + \tau ')'' (' - 1 + \tau ')'' (' - 3 + \tau ')', -27' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', -54' (' 1 + \tau ')'' (' - 3 + \tau ')', -108' (' - 1 + \tau ')', 27' (' 1 + \tau ')'' (' - 1 + \tau ')'' (' - 3 + \tau ')'' ]'$$

For τ=1/2, [60, 15, 49, 60, 16, 15] . FixedPtCheck, [60, 15, 49, 60, 16, 15]

$$\det(\mathbf{A} + \tau \Delta) = 1' (' 1 + \tau ')'' (' - 1 + \tau ')'^4$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	3 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 1/4, 0, 0, 3/4], [1/4, 0, 3/4, 0, 0, 0], [1/4, 0, 0, 3/4, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] = \\ & \$ [ [31/441, -892/3969, -1448/1701, 52352/35721, 48896/35721, -64000/35721], [55/441, 1100/3969, -56/243, -42688/35721, -13312/35721, 51200/35721], [31/441, -388/3969, 640/1701, -36352/35721, -39808/35721, 65024/35721], [85/441, 566/3969, 1504/1701, -45568/35721, -38656/35721, 41984/35721], [-95/441, -262/3969, -368/1701, 18080/35721, 32768/35721, -31744/35721], [-65/441, 548/3969, -296/1701, 65600/35721, 15488/35721, -73216/35721] ] \times \$ [ [3, 9/2, 9/2, 9/2, 6, 9/2], [15/4, 9/4, 45/8, 21/4, 27/4, 27/8], [63/16, 45/16, 51/8, 129/32, 189/32, 63/16], [519/128, 189/64, 87/16, 297/64, 441/64, 387/128], [1965/512, 1557/512, 405/64, 1029/256, 1611/256, 891/256], [2061/512, 5895/2048, 2931/512, 2217/512, 14391/2048, 3087/1024] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 1 . order: 3

$$[y_3, 0, y_1, y_2, 0, 0]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [1/81, 10/81, -8/81], [1/81, 10/81, -8/81], [10/81, -8/81, 1/81], [-8/81, 1/81, 10/81], [10/81, -8/81, 1/81], [10/81, -8/81, 1/81] ] \$ \times \$ [ [12, 0, 6, 9, 0, 0], [6, 0, 9, 12, 0, 0], [9, 0, 12, 6, 0, 0] ] \$$$

Omega Rank for B : cycles: {{4, 6}, {3, 5}}, net cycles: 1 . order: 2

$$\$ [ [0, 6, 4, 3, 8, 6], [0, 0, 8, 6, 10, 3], [0, 0, 10, 3, 8, 6], [0, 0, 8, 6, 10, 3], [0, 0, 10, 3, 8, 6] ] \$$$

$$[0, 2 y_3, -2 y_3 - 4 y_2 + 4 y_1, 2 y_2, 2 y_1, -4 y_2 + 3 y_1]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

Â» SYNC'D 27/512 , 0.05273437500

21 . Coloring, {2, 4, 6}

**R:** [4, 4, 1, 3, 3, 4] **B:** [2, 5, 5, 6, 1, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '54' ('3 + \tau^2')^2, -27' ('3 + \tau^2')^2 ('-1 + \tau')', 27' ('15 + 2\tau - 2\tau^3 + \tau^4')'' ('1 + \tau')', -54' ('3 + \tau^2')'' ('1 + \tau')'' ('-3 + \tau')', 108' ('-3 - \tau - \tau^2 + \tau^3')'' ('-1 + \tau')', 27' ('3 + \tau^2')'' ('1 + \tau')'' ('-1 + \tau')'' ('-3 + \tau')' ]'$$

For τ=1/2, [676, 169, 759, 780, 232, 195] . FixedPtCheck, [676, 169, 759, 780, 232, 195]

$$\det(A + \tau \Delta) = 1' ('1 + \tau')^2 ('-1 + \tau')^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	4 vs 4

bi =

$\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ =$

$\$ [ [-30859/1068129, 105470/1068129, 3764416/9613161, -4756736/28839483, -6734336/28839483, -748544/28839483], [-182543/2136258, -508475/2136258, -4185536/9613161, 8211688/28839483, -2968160/28839483, 17709952/28839483], [3782/1068129, -1044019/1068129, -21323984/9613161, -20189936/28839483, 16861504/28839483, 96454912/28839483], [553477/2136258, 1296097/2136258, 16201480/9613161, 18806920/28839483, -11245280/28839483, -80067200/28839483], [-115306/1068129, -406891/1068129, -7252232/9613161, -9359696/28839483, 23552320/28839483, 22731520/28839483], [205987/2136258, 2059507/2136258, 9463360/9613161, 9817432/28839483, -20578208/28839483, -47145344/28839483] ] \$ \times \$ [ [13/2, 9/2, 5/2, 3, 6, 9/2], [17/2, 39/8, 9/4, 31/8, 21/4, 9/4], [99/16, 51/8, 73/32, 125/32, 171/32, 93/32], [865/128, 297/64, 37/16, 495/128, 831/128, 375/128], [1957/256, 2595/512, 663/256, 917/256, 1335/256, 1485/512], [13791/2048, 5871/1024, 563/256, 3997/1024, 11763/2048, 2751/1024] ] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 1 . order: 3

$[y_3, 0, y_1, y_2, 0, 0]$

$R = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [94/1053, -95/1053, 40/1053], [94/1053, -95/1053, 40/1053], [-95/1053, 40/1053, 94/1053], [40/1053, 94/1053, -95/1053], [40/1053, 94/1053, -95/1053], [94/1053, -95/1053, 40/1053] ] \$ \times \$ [ [5, 0, 10, 12, 0, 0], [10, 0, 12, 5, 0, 0], [12, 0, 5, 10, 0, 0] ] \$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$[y_2, y_3, 0, 0, y_1, y_4]$

$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, -35/1539, 154/1539, -62/1539], [0, -62/1539, -35/1539, 154/1539], [0, -62/1539, -35/1539, 154/1539], [1/6, -62/1539, -35/1539, -205/3078], [0, 154/1539, -62/1539, -35/1539], [0, 154/1539, -62/1539, -35/1539] ] \$ \times \$ [ [7, 6, 0, 0, 8, 6], [14, 7, 0, 0, 6, 0], [6, 14, 0, 0, 7, 0], [7, 6, 0, 0, 14, 0] ] \$$

Â» SYNC'D 41/128 , 0.3203125000

22 . Coloring, {2, 5, 6}

**R:** [4, 4, 1, 6, 1, 4] **B:** [2, 5, 5, 3, 3, 1]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\left[ \begin{aligned} &54 \tau^2 (\tau - 1 + \tau^2)^2 (\tau^3 + \tau^2)^2 (\tau + 1)^2 (\tau - 3 + \tau^2)^2, -27 \tau^2 (\tau - 1 + \tau^2)^2 (\tau^3 + \tau^2)^2 (\tau + 1)^2 \\ &(\tau - 3 + \tau^2)^2, 27 \tau^2 (\tau - 1 + \tau^2)^2 (\tau^3 + \tau^2)^2 (\tau + 1)^2 (\tau - 3 + \tau^2)^2, 54 \tau^2 (\tau - 1 + \tau^2)^2 (\tau^3 + \tau^2)^2 \\ &(\tau - 3 + \tau^2)^2, -108 \tau^2 (\tau - 1 + \tau^2)^2 (\tau^3 + \tau^2)^2, 27 \tau^2 (\tau - 1 + \tau^2)^2 (\tau^3 + \tau^2)^2 (\tau + 1)^2 \end{aligned} \right]$$

For  $\tau=1/2$ , [420, 105, 247, 900, 88, 675] . FixedPtCheck, [420, 105, 247, 900, 88, 675]

$$\det(A + \tau \Delta) = \tau^3 (\tau - 1 + \tau^2)^3 (\tau + 1)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	3 vs 3	4 vs 4

bi =

$$\begin{aligned} &\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 3/4, 0, 0, 1/4], [1/4, 0, 3/4, 0, 0, 0], \\ &[3/4, 0, 0, 1/4, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ &\$ [ [12/29, -7/87, -808/261, -656/261, 2368/783, 1792/783], [-41/406, 761/1218, -524/1827, \\ &-13864/5481, -1184/5481, 13952/5481], [-46/609, -181/1827, 968/1827, 752/609, -4160/5481, \\ &-4352/5481], [25/406, -365/1218, 436/203, -4840/1827, -20192/5481, 24448/5481], [-67/609, -118/1827, \\ &1304/1827, 2368/1827, -128/5481, -9728/5481], [-99/406, 157/406, 212/609, 16568/1827, 15520/5481, \\ &-67712/5481] ] \$ \times \$ [ [9/2, 9/2, 15/2, 3, 6, 3/2], [9/2, 27/8, 27/4, 21/8, 9, 3/4], [9/2, 27/8, 279/32, 69/32, \\ &243/32, 21/32], [585/128, 27/8, 117/16, 273/128, 1161/128, 69/128], [9/2, 1755/512, 2151/256, 543/256, \\ &513/64, 273/512], [9225/2048, 27/8, 7785/1024, 1083/512, 18171/2048, 543/1024] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 2

$$[y_2, 0, 0, y_1, 0, y_3]$$

$$\begin{aligned} R = &\$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \\ &\times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \\ &= \$ [ [0, 5/27, -4/27], [0, 5/27, -4/27], [1/9, -4/27, 2/27], [0, -4/27, 5/27], [1/9, -4/27, 2/27], \\ &[0, 5/27, -4/27] ] \times \$ [ [9, 0, 0, 12, 0, 6], [0, 0, 0, 15, 0, 12], [0, 0, 0, 12, 0, 15] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{3, 5}}, net cycles: 0 . order: 4

$$[y_4, y_3, y_2, 0, y_1, 0]$$

$$\begin{aligned} B = &\$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0] ] \\ &\times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \\ &= \$ [ [0, 1/3, 16/135, -56/135], [0, 0, -11/135, 16/135], [0, 0, -11/135, 16/135], [0, 0, 16/135, 0, 0, 0] ] \$ \end{aligned}$$

$-11/135], [0, 0, 16/135, -11/135], [1/3, -2/3, -56/135, 106/135] ] \times [ [3, 6, 10, 0, 8, 0], [0, 3, 8, 0, 16, 0], [0, 0, 16, 0, 11, 0], [0, 0, 11, 0, 16, 0] ]$

Â» SYNC'D 3/32 , 0.09375000000

23 . Coloring, {3, 4, 5}

**R:** [4, 5, 5, 3, 1, 1] **B:** [2, 4, 1, 6, 3, 4]

' See graph

' ' See pair graph

'

Ω for  $A+\tau\Delta$  :

$[ '54' ( '1 + \tau' )'' ( '3 + \tau^2' )'' ( '-3 + \tau' )', -27' ( '-1 + \tau' )'' ( '1 + \tau' )'' ( '3 + \tau^2' )'' ( '-3 + \tau' )', 27' ( '1 + \tau' )'' ( '-15 + 4\tau - 2\tau^2 - 4\tau^3 + \tau^4' )', -54' ( '3 + \tau^2' )'^2, -108' ( '1 + \tau' )'^2 ( '3 - 2\tau + \tau^2' )', 27' ( '-1 + \tau' )'' ( '3 + \tau^2' )'^2 ]'$

For  $\tau=1/2$ , [-780, -195, -669, -676, -648, -169] . FixedPtCheck, [780, 195, 669, 676, 648, 169]

$\det(A + \tau \Delta) = 1' ( '-1 + \tau' )'^4 ( '1 + \tau' )'$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 1/4, 0, 0, 3/4], [1/4, 0, 3/4, 0, 0, 0], [1/4, 0, 0, 3/4, 0, 0] ] \times [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ]$  =

$\$ [ [9066/98321, 19579/294963, 202496/2654667, -3823024/2654667, -334528/2654667, 3632384/2654667], [-17189/589926, -275569/5309334, 1367960/2654667, -85384/2654667, -2669344/2654667, 1700224/2654667], [-36768/98321, 863099/884889, -2281936/2654667, 5217296/2654667, 4884032/2654667, -9317632/2654667], [-67945/196642, 672919/1769778, -277456/2654667, 3109352/2654667, 3613088/2654667, -6438784/2654667], [81993/98321, -130254/98321, 1742360/2654667, -2899840/2654667, -6080128/2654667, 8638976/2654667], [74815/196642, -3719491/5309334, 262040/2654667, -3316312/2654667, -3920992/2654667, 7923328/2654667] ] \times [ [11/2, 9/2, 9/2, 6, 2, 9/2], [5, 33/8, 3, 65/8, 9/4, 9/2], [63/16, 15/4, 119/32, 247/32, 57/32, 195/32], [609/128, 189/64, 209/64, 1071/128, 239/128, 741/128], [1117/256, 1827/512, 447/128, 1983/256, 199/128, 3213/512], [9373/2048, 3351/1024, 3177/1024, 8677/1024, 3615/2048, 5949/1024] ]$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4, 5}}, net cycles: 1 . order: 4

$$[y_4, 0, y_1, y_2, y_3, 0]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [8/135, -19/135, 62/135, -46/135], [62/135, -46/135, 8/135, -19/135], [62/135, -46/135, 8/135, -19/135], [-46/135, 8/135, -19/135, 62/135], [-19/135, 62/135, -46/135, 8/135], [-19/135, 62/135, -46/135, 8/135] ] \$ \times \$ [ [7, 0, 6, 6, 8, 0], [8, 0, 6, 7, 6, 0], [6, 0, 7, 8, 6, 0], [6, 0, 8, 6, 7, 0] ] \$$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0, y_5]$$

$$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 1/4, -11/135, -71/540], [0, 0, 0, 16/135, -11/135], [0, 1/4, -5/16, -71/540, 499/2160], [0, 0, 0, -11/135, 16/135], [1/4, -5/16, 1/64, 499/2160, -1271/8640], [0, 0, 0, 16/135, -11/135] ] \$ \times \$ [ [5, 6, 4, 6, 0, 6], [4, 5, 0, 12, 0, 6], [0, 4, 0, 11, 0, 12], [0, 0, 0, 16, 0, 11], [0, 0, 0, 11, 0, 16] ] \$$$

Â» SYNC'D 343/4096 , 0.08374023438

24 . Coloring, {3, 4, 6}

**R:** [4, 5, 5, 3, 3, 4] **B:** [2, 4, 1, 6, 1, 1]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '-54' (' 3 + \tau ')'' (' - 1 + \tau ')', 27' (' 3 + \tau ')'' (' - 1 + \tau ')'^ 2, -27' (' 1 + \tau ')'' (' - 5 + \tau ^ 2 ')', -54' (' 3 + \tau ')'' (' - 1 + \tau ')', 108' (' 1 + \tau ')', 27' (' 3 + \tau ')'' (' - 1 + \tau ')'^ 2 ]'$$

For τ=1/2, [28, 7, 57, 28, 48, 7] . FixedPtCheck, [28, 7, 57, 28, 48, 7]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')'^ 2 ' (' - 1 + \tau ')'^ 3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	4 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [3/4, 0, 0, 1/4, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-121/2613, 64/2613, 6328/7839, -51008/70551, 8960/7839, -82432/70551], [83/2613, \\ & 388/2613, -25576/23517, 21184/23517, -5120/2613, 47104/23517], [445/871, -984/871, -4096/2613, \\ & -3136/23517, -41344/23517, 3584/871], [73/2613, -4622/7839, 3200/2613, -23680/23517, 42752/23517, \\ & -11264/7839], [-257/871, 1590/871, 1520/2613, 30560/23517, 3584/23517, -3072/871], [-313/2613, \\ & 3376/7839, -26792/23517, 73984/70551, -29056/23517, 74240/70551] ] \$ \times \$ [ [9, 9/2, 5/2, 9/2, 2, 9/2], \\ & [27/4, 27/4, 13/8, 27/4, 7/4, 27/8], [81/16, 81/16, 17/8, 243/32, 67/32, 81/16], [891/128, 243/64, 155/64, \\ & 405/64, 115/64, 729/128], [3807/512, 2673/512, 65/32, 1539/256, 199/128, 1215/256], [6399/1024, \\ & 11421/2048, 1937/1024, 891/128, 3713/2048, 4617/1024] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 5}}, net cycles: 0 . order: 2

$$[0, 0, y_1, y_2, y_3, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0, \\ & 0] ] \times \$ [ [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, \\ & 0, 0, 0] ] \$ = & \$ [ [1/9, -10/189, -4/189], [0, -10/189, 17/189], [0, -10/189, 17/189], [0, 17/189, -10/189], \\ & [0, 17/189, -10/189], [1/9, -10/189, -4/189] ] \times \$ [ [0, 0, 10, 9, 8, 0], [0, 0, 17, 0, 10, 0], [0, 0, 10, 0, 17, \\ & 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 4, 6}}, net cycles: 1 . order: 4

$$[y_2, y_3, 0, y_1, 0, y_4]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, \\ & 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, \\ & 0, 0, 1] ] \$ = & \$ [ [-2/27, 4/27, -2/27, 1/27], [1/27, -2/27, 4/27, -2/27], [4/27, -2/27, 1/27, -2/27], [-2/27, \\ & 1/27, -2/27, 4/27], [4/27, -2/27, 1/27, -2/27], [4/27, -2/27, 1/27, -2/27] ] \times \$ [ [12, 6, 0, 3, 0, 6], [6, 12, \\ & 0, 6, 0, 3], [3, 6, 0, 12, 0, 6], [6, 3, 0, 6, 0, 12] ] \$ \end{aligned}$$

Â» SYNC'D 75/256 , 0.2929687500

25 . Coloring, {3, 5, 6}

**R:** [4, 5, 5, 6, 1, 4]   **B:** [2, 4, 1, 3, 3, 1]

‘ See graph

‘ ‘ See pair graph



$\Omega$  for  $A+\tau\Delta$  :

$$\left[ \begin{array}{c} -54(\tau^3 + \tau^2) \tau^2 (-1 + \tau)^2, 27(\tau^3 + \tau^2) \tau^2 (-1 + \tau)^2, -27(\tau^3 + \tau^2) \tau^2 (-1 + \tau)^2, 54(\tau^3 + \tau^2) \tau^2 (-1 + \tau)^2, 108(\tau^3 + \tau^2) \tau^2 (-1 + \tau)^2, 27(\tau^3 + \tau^2) \tau^2 (-1 + \tau)^2 \end{array} \right]$$

For  $\tau=1/2$ , [364, 91, 229, 676, 240, 507] . FixedPtCheck, [364, 91, 229, 676, 240, 507]

$$\det(A + \tau \Delta) = 1^3 (-1 + \tau)^3 (1 + \tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 3/4, 0, 0, 1/4], [1/4, 0, 3/4, 0, 0, 0], [3/4, 0, 0, 1/4, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\$ [ [441595/14243773, 77691556/128193957, 64095112/128193957, -26893504/128193957, 497408/384581871, -342856192/384581871], [268003/14243773, -32142212/128193957, 31816/128193957, 29116544/128193957, -272215552/384581871, 288204800/384581871], [-2099037/14243773, -48917420/128193957, -72175184/128193957, -14521600/128193957, 117610880/384581871, 360149504/384581871], [535273/14243773, 105627646/128193957, 69600016/128193957, 61888256/128193957, 151068416/384581871, -862624768/384581871], [-2731619/14243773, -301605830/128193957, -134724896/128193957, -95473504/128193957, 184398080/384581871, 1499012096/384581871], [9666739/14243773, 149173948/128193957, 32503096/128193957, 52394624/128193957, -472798336/384581871, -476174848/384581871] ] \times \$ [ [7, 9/2, 15/2, 9/2, 2, 3/2], [29/4, 21/4, 39/8, 11/2, 3, 9/8], [21/4, 87/16, 51/8, 193/32, 81/32, 11/8], [825/128, 63/16, 411/64, 367/64, 189/64, 193/128], [3423/512, 2475/512, 417/64, 1265/256, 663/256, 367/256], [423/64, 10269/2048, 723/128, 5791/1024, 5811/2048, 1265/1024] ] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, 0, 0, y_2, y_3, y_4]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 17/189, -10/189], [1/8, -1/16, -53/1512, 29/3024], [1/8, -1/16, -53/1512, 29/3024], [0, 0, -10/189, 17/189], [0, 1/8, -10/189, -53/1512], [0, 0, 17/189, -10/189] ] \times \$ [ [4, 0, 0, 9, 8, 6], [8, 0, 0, 10, 0, 9], [0, 0, 0, 17, 0, 10], [0, 0, 0, 10, 0, 17] ] \$$$

Omega Rank for B : cycles: {{1, 2, 3, 4}}, net cycles: 1 . order: 4

$$[y_1, y_2, y_3, y_4, 0, 0]$$

$$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [58/5535, 112/5535, 598/5535, -563/5535], [-563/5535, 58/5535, 112/5535, 598/5535], [112/5535, 598/5535, -563/5535, 58/5535], [598/5535, -563/5535, 58/5535, 112/5535], [598/5535, -563/5535, 58/5535, 112/5535], [112/5535, 598/5535, -563/5535, 58/5535] ] \$ \times \$ [ [8, 6, 10, 3, 0, 0], [10, 8, 3, 6, 0, 0], [3, 10, 6, 8, 0, 0], [6, 3, 8, 10, 0, 0] ] \$$$

Â» SYNC'D 231/1024 , 0.2255859375

26 . Coloring, {4, 5, 6}

**R:** [4, 5, 1, 3, 1, 4]    **B:** [2, 4, 5, 6, 3, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '-18' (' - 3 + \tau ')', 9' (' - 1 + \tau ')'' (' - 3 + \tau ')', 9' (' 5 - 2\tau + \tau^2 ')', -18' (' - 3 + \tau ')', -36' (' - 1 + \tau ')', 9' (' - 1 + \tau ')'' (' - 3 + \tau ')'' ]'$$

For τ=1/2, [20, 5, 17, 20, 8, 5] . FixedPtCheck, [20, 5, 17, 20, 8, 5]

$$\det(A + \tau \Delta) = 1' (' 1 + 3\tau^2 ')'' (' - 1 + \tau ')'^3$$

Delta Range : [-y<sub>1</sub> - y<sub>2</sub> - y<sub>3</sub> - y<sub>4</sub> - y<sub>5</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>]

$$[6, 3, 5, 6, 4, 3]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\$ [ [9, 0, 6, 9, 3, 0], [15, 3, 14, 15, 4, 3], [27, 9, 27, 27, 9, 9], [51, 21, 50, 51, 22, 21], [99, 45, 93, 99, 51, 45] ] \$ \quad \$ [ [3, 6, 4, 3, 5, 6], [9, 9, 6, 9, 12, 9], [21, 15, 13, 21, 23, 15], [45, 27, 30, 45, 42, 27], [93, 51, 67, 93, 77, 51] ] \$ \quad \$ [ [3, -3, 1, 3, -1, -3], [3, -3, 4, 3, -4, -3], [3, -3, 7, 3, -7, -3], [3, -3, 10, 3, -10, -3], [3, -3, 13, 3, -13, -3] ] \$$$

$$[-y_2, y_2, -y_1, -y_2, y_1, y_2]$$

$$p = s - 32s^4 + 48s^5$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$$\$ [ [7, 11, 16, 27, 14, 6], [12, 0, 14, 18, 20, 17], [24, 10, 10, 20, 7, 10], [23, 6, 19, 7, 15, 11], [24, 10, 10, 20, 7, 10], [18, 17, 21, 16, 9, 0] ] \$ \quad \$ [ [24, 14, 10, 14, 9, 10], [17, 18, 17, 20, 6, 3], [17, 6, 21, 13, 17, 7], [13, 7, 14, 27, 10, 10], [17, 6, 21, 13, 17, 7], [20, 3, 7, 21, 13, 17] ] \$ \quad \$ [ [36, 12, 15, 18, 12, 15], [24, 18, 15, 30, 12, 9], [18, 9, 30, 18, 24, 9], [18, 15, 15, 36, 12, 12], [18, 9, 30, 18, 24, 9], [30, 9, 15, 24,$$

12, 18] ] \$

CmmCk true, true, true

$$p' = s - 12s^3 + 16s^4 \quad p' = s^2 - 4s^3 + 4s^4$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
2 vs 5	3 vs 6	3 vs 6	2 vs 4	2 vs 6

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 0 . order: 3

$$\$ [ [9, 0, 6, 9, 3, 0], [9, 0, 9, 9, 0, 0], [9, 0, 9, 9, 0, 0], [9, 0, 9, 9, 0, 0] ] \$$$

$$[y_1, 0, y_2, y_1, -y_2 + y_1, 0]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{1, 2, 4, 6}, {3, 5}}, net cycles: 2 . order: 4

$$\$ [ [3, 6, 4, 3, 5, 6], [6, 3, 5, 6, 4, 3], [3, 6, 4, 3, 5, 6], [6, 3, 5, 6, 4, 3], [3, 6, 4, 3, 5, 6], [6, 3, 5, 6, 4, 3] ] \$$$

$$[-2y_2 + 3y_1, y_2, -y_2 + 2y_1, -2y_2 + 3y_1, y_1, y_2]$$

$$p' = -1 + s^4 \quad p' = -s + s^5 \quad p' = -s + s^3 \quad p' = -1 + s^2$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, x_2, x_3, -32x_1 - 12x_2 - 4x_3, 48x_1 + 16x_2 + 4x_3]$$

$$\text{For } A+2\Delta : [-y_1 - y_2, -y_2 - y_3, y_2, y_1, y_2, y_3]$$

$$\text{For } A-2\Delta : [-y_1 - y_2, -y_2 - y_3, y_2, y_1, y_2, y_3]$$

Range of  $\{\Omega\Delta^i\}$ :  $[-\mu_2, \mu_2, -\mu_1, -\mu_2, \mu_1, \mu_2]$

rank of M is 6 , rank of N is 4

M          N

$$\$ [ [0, 0, 4, 6, 2, 0], [0, 0, 1, 0, 2, 3], [4, 1, 0, 4, 0, 1], [6, 0, 4, 0, 2, 0], [2, 2, 0, 2, 0, 2], [0, 3, 1, 0, 2, 0] ]$$

$$\$ \quad \$ [ [0, 2, 3, 3, 3, 1], [2, 0, 3, 1, 3, 3], [3, 3, 0, 3, 0, 3], [3, 1, 3, 0, 3, 2], [3, 3, 0, 3, 0, 3], [1, 3, 3, 2, 3, 0] ] \$$$

Check is  $\Omega\Delta N$  zero? true,  $\pi\Delta = [3, -3, 1, 3, -1, -3]$

ker M,  $[0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_6, x_2, x_3, x_4, x_5]$

$$\tau = 12, r' = 2/3$$

Ranges

Action of R on ranges, [[1], [1], [2], [2]]

Action of B on ranges, [[4], [3], [2], [1]]

$$\beta(\{1, 3, 4\}) = 4/9$$

$$\beta(\{1, 4, 5\}) = 2/9$$

$$\beta(\{2, 3, 6\}) = 1/9$$

$$\beta(\{2, 5, 6\}) = 2/9$$

ker N,  $[\mu_1, -\mu_1, -\mu_2, \mu_1, \mu_2, -\mu_1]$

Range of N

$$[y_2, y_3, y_4, -y_2 + y_3 + y_1, y_4, y_1]$$

Partitions

Action of R on partitions, [[1], [1]]

Action of B on partitions, [[2], [1]]

$$\alpha(\{\{2, 4\}, \{3, 5\}, \{1, 6\}\}) = 2/3$$

$$\alpha(\{\{1, 2\}, \{3, 5\}, \{4, 6\}\}) = 1/3$$

$$b_1 = \{1, 2\}, b_2 = \{2, 4\}, b_3 = \{3, 5\}, b_4 = \{4, 6\}, b_5 = \{1, 6\}$$

Action of R and B on the blocks of the partitions:  $[0, 0, 1, 0, 1], [1, 0, 0, 0, 1], [0, 1, 1, 0, 0], [0, 1, 0, 0, 1], [0, 0, 1, 1, 0]$   $= [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0]$   $+ [0, 0, 0, 0, 1], [1, 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0], [0, 0, 0, 1, 0]$   $[3', 5', 2', 5', 3'], [5', 1', 3', 2', 4']$  with invariant measure  $[1, 2, 3, 1, 2]$

N by blocks, check: true . ' See partition graph.

' ' See level-3 partition graph.

'

Sandwich	
<b>Coloring</b>	{4, 5, 6}
<b>Rank</b>	3
<b>R,B</b>	[4, 5, 1, 3, 1, 4], [2, 4, 5, 6, 3, 1]
$\pi_2$	[0, 4, 6, 2, 0, 1, 0, 2, 3, 4, 0, 1, 2, 0, 2]
$u_2$	[2, 3, 3, 3, 1, 3, 1, 3, 3, 3, 0, 3, 3, 2, 3] (dim 1)
<b>wpp</b>	[2, 2, 2, 2, 2, 2]
$\pi_3$	[0, 0, 0, 0, 4, 0, 0, 2, 0, 0, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0]
$u_3$	[2, 0, 2, 0, 3, 0, 1, 3, 0, 1, 1, 0, 3, 1, 0, 3, 0, 2, 0, 2]

27 . Coloring, {2, 3, 4, 5}

**R:** [4, 4, 5, 3, 1, 1] **B:** [2, 5, 1, 6, 3, 4]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘ -54‘ (‘ 3 +  $\tau^2$  ‘)‘ , 27‘ (‘ 3 +  $\tau^2$  ‘)‘ ‘ (‘ - 1 +  $\tau$  ‘)‘ , 27‘ (‘ - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  ‘)‘ , -54‘ (‘ 3 +  $\tau^2$  ‘)‘ , -108‘ (‘ 1 +  $\tau^2$  ‘)‘ , 27‘ (‘ 3 +  $\tau^2$  ‘)‘ ‘ (‘ - 1 +  $\tau$  ‘)‘]‘

For  $\tau=1/2$ , [-52, -13, -49, -52, -40, -13] . FixedPtCheck, [52, 13, 49, 52, 40, 13]

$\det(A + \tau \Delta) = 1 \cdot (1 + 3\tau^2) \cdot (-1 + \tau)^3$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 6

bi =

\$ [ [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 0, 1/4, 3/4, 0] , [3/4, 0, 0, 0, 1/4, 0] , [0, 0, 1/4, 0, 0, 3/4] , [1/4, 0, 3/4, 0, 0, 0] , [1/4, 0, 0, 3/4, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ =

\$ [ [436/1481, 105473/93303, 47144/279909, -83344/279909, 17408/279909, -369664/279909] , [1050/1481, 50069/31101, 457616/279909, 120704/279909, -726784/279909, -490240/279909] ,

[1785/1481, 122210/93303, 245972/279909, 352112/279909, -647872/279909, -643840/279909] ,  
 [-1706/1481, -298873/93303, -528820/279909, -520096/279909, 737984/279909, 1540352/279909] ,  
 [1207/1481, 206336/93303, 590708/279909, 630320/279909, -575296/279909, -1482496/279909] ,  
 [-7802/4443, -80735/31101, -691828/279909, -341104/279909, 1062848/279909, 1198592/279909] ] \$ x  
 \$ [ [11/2, 9/2, 9/2, 9/2, 7/2, 9/2] , [43/8, 33/8, 15/4, 47/8, 9/2, 27/8] , [153/32, 129/32, 155/32, 157/32,  
 129/32, 141/32] , [735/128, 459/128, 17/4, 705/128, 271/64, 471/128] , [2645/512, 2205/512, 2331/512,  
 2607/512, 1921/512, 2115/512] , [11029/2048, 7935/2048, 4185/1024, 11195/2048, 4473/1024,  
 7821/2048] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4, 5}}, net cycles: 1 . order: 4

$$[y_1, 0, y_2, y_3, y_4, 0]$$

R = \$ [ [0, 0, 0, 1, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0,  
 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0,  
 0, 0, 0] ] \$ = \$ [ [173/459, -97/459, 65/459, -124/459] , [173/459, -97/459, 65/459, -124/459] , [65/459,  
 -124/459, 173/459, -97/459] , [-124/459, 173/459, -97/459, 65/459] , [-97/459, 65/459, -124/459, 173/459]  
 , [-97/459, 65/459, -124/459, 173/459] ] \$ x \$ [ [7, 0, 6, 9, 5, 0] , [5, 0, 9, 7, 6, 0] , [6, 0, 7, 5, 9, 0] , [9, 0,  
 5, 6, 7, 0] ] \$

Omega Rank for B : cycles: {{1, 2, 3, 5}, {4, 6}}, net cycles: 2 . order: 4

\$ [ [5, 6, 4, 3, 3, 6] , [4, 5, 3, 6, 6, 3] , [3, 4, 6, 3, 5, 6] , [6, 3, 5, 6, 4, 3] , [5, 6, 4, 3, 3, 6] , [4, 5, 3, 6, 6, 3] ]  
 \$

$$[2y_4, 2y_3, 4y_4 - 2y_3 - 4y_2 + 4y_1, 2y_2, 2y_1, 3y_4 - 4y_2 + 3y_1]$$

$$p' = -s + s^5 \quad p' = -1 + s^4$$

Â» SYNC'D 479/16384 , 0.02923583984

28 . Coloring, {2, 3, 4, 6}

**R:** [4, 4, 5, 3, 3, 4] **B:** [2, 5, 1, 6, 1, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

[ '-54' (' - 1 + τ ')'' (' 3 + τ ')'' (' 3 + τ <sup>2</sup> ')', 27' (' - 1 + τ ')'' <sup>2</sup> (' 3 + τ ')'' (' 3 + τ <sup>2</sup> ')', 27'  
 (' 1 + τ ')'' (' 15 + 2τ - 2τ <sup>3</sup> + τ <sup>4</sup> ')', 54' (' - 1 + τ ')'' (' 3 + τ ')'' (' 1 + τ ')'' (' - 3 + τ ')', -108' ('  
 - 3 - τ - 5τ <sup>2</sup> + τ <sup>3</sup> ')', -27' (' - 1 + τ ')'' <sup>2</sup> (' 3 + τ ')'' (' 1 + τ ')'' (' - 3 + τ ')'' ]'

For  $\tau=1/2$ , [364, 91, 759, 420, 592, 105] . FixedPtCheck, [364, 91, 759, 420, 592, 105]

$$\det(A + \tau \Delta) = 1^4 (-1 + \tau)^3 (-1 + \tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	3 vs 3	4 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 1/4, 0, 0, 3/4], [3/4, 0, 1/4, 0, 0, 0], \\ & [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-3460/91653, 377/30551, 9320/91653, -223024/824877, 217024/824877, -26368/824877], \\ & [-6601/91653, -5700/30551, -122900/274959, 2288/30551, -242048/824877, 792832/824877], \\ & [2754/30551, -16317/30551, -131744/91653, -55024/274959, 25408/824877, 1722112/824877], \\ & [6704/30551, 38387/91653, 124540/91653, 28640/274959, -192320/824877, -1510400/824877], \\ & [-3267/30551, -10296/30551, -35408/91653, 41312/274959, 410752/824877, 180736/824877], \\ & [3146/91653, 60833/91653, 120092/274959, 322304/824877, -397376/824877, -830464/824877] ] \$ \times \$ \\ & [ [9, 9/2, 5/2, 3, 7/2, 9/2], [63/8, 27/4, 13/8, 9/2, 4, 9/4], [189/32, 189/32, 17/8, 135/32, 175/32, 27/8], \\ & [1053/128, 567/128, 155/64, 243/64, 635/128, 405/128], [2025/256, 3159/512, 1121/512, 2025/512, \\ & 2011/512, 729/256], [6885/1024, 6075/1024, 1009/512, 8667/2048, 5299/1024, 6075/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 5}}, net cycles: 0 . order: 2

$$[0, 0, y_2, y_3, y_1, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [1/12, -10/189, 5/756], [1/12, -10/189, 5/756], [0, -10/189, 17/189], [0, 17/189, -10/189], \\ & [0, 17/189, -10/189], [1/12, -10/189, 5/756] ] \$ \times \$ [ [0, 0, 10, 12, 5, 0], [0, 0, 17, 0, 10, 0], [0, 0, 10, 0, 17, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_1, y_4, 0, 0, y_2, y_3]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], \\ & [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 10/81, 1/81, -8/81], [0, -8/81, 10/81, 1/81], [0, 1/81, -8/81, 10/81], [1/6, -8/81, 10/81, -25/162], \\ & [0, 1/81, -8/81, 10/81], [0, 1/81, -8/81, 10/81] ] \$ \times \$ [ [12, 6, 0, 0, 3, 6], [9, 12, 0, 0, 6, 0], [6, 9, 0, 0, 12, 0], [12, 6, 0, 0, 9, 0] ] \$ \end{aligned}$$

$\hat{A}$ » SYNC'D 25/64 , 0.3906250000

29 . Coloring, {2, 3, 5, 6}

**R:** [4, 4, 5, 6, 1, 4] **B:** [2, 5, 1, 3, 3, 1]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & \text{' [ ' -54' ( ' 3 + \tau ' )'' ( ' - 1 + \tau ' )' , 27' ( ' 3 + \tau ' )'' ( ' - 1 + \tau ' )' ^ 2 , 27' ( ' - 5 + \tau ^ 2 ' )'' ( ' - 1 + \tau ' )' , } \\ & \text{-54' ( ' 1 + \tau ' )'' ( ' - 3 + \tau ' )' , -108' ( ' - 1 + \tau ' )' , -27' ( ' 1 + \tau ' )' ^ 2 ( ' - 3 + \tau ' )' ]' } \end{aligned}$$

For  $\tau=1/2$ , [28, 7, 19, 60, 16, 45] . FixedPtCheck, [28, 7, 19, 60, 16, 45]

$$\det(A + \tau \Delta) = 1' ( ' - 1 + \tau ' )' ^ 3 ( ' 1 + \tau ' )' ^ 2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 0, 1/4, 3/4, 0] , [3/4, 0, 0, 0, 1/4, 0] , [0, 0, 3/4, 0, 0, 1/4] , [1/4, 0, 3/4, 0, 0, 0] , \\ & [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [ [156746/9003137, -100799/2455401, 1702400/9003137, -35129360/81028233, \\ & -25531712/243084699, 99705088/243084699] , [-1698847/9003137, 1249678/2455401, \\ & 4243364/9003137, 48941488/81028233, -43238528/243084699, -287002880/243084699] , \\ & [352194/9003137, -364295/2455401, -4543312/9003137, -8172848/81028233, -2874176/243084699, \\ & 185621248/243084699] , [4707068/9003137, -3613445/2455401, -7457612/9003137, \\ & -41917088/81028233, -192131648/243084699, 758881792/243084699] , [-1195321/9003137, \\ & 620692/2455401, 2117912/9003137, 23038144/81028233, 164142208/243084699, \\ & -310611968/243084699] , [-4020964/9003137, 5958379/2455401, 12015364/9003137, \\ & 88055296/81028233, 264499264/243084699, -1325390336/243084699] ] \$ \times \$ [ [7, 9/2, 15/2, 3, 7/2, \\ & 3/2] , [61/8, 21/4, 39/8, 13/4, 21/4, 3/4] , [177/32, 183/32, 51/8, 109/32, 165/32, 13/16] , [855/128, \\ & 531/128, 411/64, 193/64, 753/128, 109/128] , [1773/256, 2565/512, 3417/512, 1495/512, 2415/512, \\ & 193/256] , [27/4, 5319/1024, 5865/1024, 6497/2048, 1389/256, 1495/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, 0, 0, y_2, y_3, y_4]$$



$$\mathbf{R} = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 17/189, -10/189], [0, 0, 17/189, -10/189], [1/5, -4/25, -104/945, 506/4725], [0, 0, -10/189, 17/189], [0, 1/5, -10/189, -104/945], [0, 0, 17/189, -10/189] ] \$ \times \$ [ [4, 0, 0, 12, 5, 6], [5, 0, 0, 10, 0, 12], [0, 0, 0, 17, 0, 10], [0, 0, 0, 10, 0, 17] ] \$$$

Omega Rank for B : cycles: {{1, 2, 3, 5}}, net cycles: 1 . order: 4

$$[y_3, y_4, y_2, 0, y_1, 0]$$

$$\mathbf{B} = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [58/5535, 112/5535, 598/5535, -563/5535], [-563/5535, 58/5535, 112/5535, 598/5535], [112/5535, 598/5535, -563/5535, 58/5535], [598/5535, -563/5535, 58/5535, 112/5535], [598/5535, -563/5535, 58/5535, 112/5535], [112/5535, 598/5535, -563/5535, 58/5535] ] \$ \times \$ [ [8, 6, 10, 0, 3, 0], [10, 8, 3, 0, 6, 0], [3, 10, 6, 0, 8, 0], [6, 3, 8, 0, 10, 0] ] \$$$

Â» SYNC'D 241/1024 , 0.2353515625

30 . Coloring, {2, 4, 5, 6}

**R:** [4, 4, 1, 3, 1, 4] **B:** [2, 5, 5, 6, 3, 1]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '54' ('3 + \tau^2')' ('1 + \tau')' ('-3 + \tau')', -27' ('-1 + \tau')' ('3 + \tau^2')' ('1 + \tau')' ('-3 + \tau')', 27' ('-15 - 11\tau - 14\tau^2 + 10\tau^3 - 3\tau^4 + \tau^5')', -54' ('1 + \tau')'^2 ('-3 + \tau')'^2, -108' ('-3 - \tau - \tau^2 + \tau^3')' ('-1 + \tau')', 27' ('-1 + \tau')' ('1 + \tau')'^2 ('-3 + \tau')'^2 ]'$$

For τ=1/2, [-780, -195, -733, -900, -232, -225] . FixedPtCheck, [780, 195, 733, 900, 232, 225]

$$\det(\mathbf{A} + \tau \Delta) = 1' ('-1 + \tau')'^4 ('1 + \tau')'$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	5 vs 5

bi =

$$\$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4, 0], [1/4, 0, 0, 0, 3/4, 0], [0, 0, 1/4, 0, 0, 3/4], [1/4, 0, 3/4, 0, 0, 0], [3/4, 0, 0, 1/4, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [-28/165, 19/1485, 848/891, -496/4455, -3136/4455, 256/4455], [-1403/11880, -18901/106920,$$

1646/8019, 16186/13365, -3944/13365, -31648/40095] , [163/1980, -199/17820, -956/2673, -1412/4455, -752/4455, 10816/13365] , [121/360, -553/3240, -130/243, 98/405, 248/405, -544/1215] , [-19/660, 7/5940, -172/891, -532/4455, 2768/4455, -1088/4455] , [47/2376, 10897/21384, -1486/8019, -2098/2673, -184/2673, 4384/8019] ] \$ x \$ [ [9/2, 9/2, 9/2, 3, 6, 9/2] , [6, 27/8, 21/4, 27/8, 27/4, 9/4] , [75/16, 9/2, 189/32, 93/32, 207/32, 81/32] , [639/128, 225/64, 357/64, 375/128, 999/128, 279/128] , [1275/256, 1917/512, 843/128, 171/64, 873/128, 1125/512] , [10239/2048, 3825/1024, 2961/512, 699/256, 15867/2048, 513/256] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 1 . order: 3

$$[y_1, 0, y_2, y_3, 0, 0]$$

R = \$ [ [0, 0, 0, 1, 0, 0] , [0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [10/81, 1/81, -8/81] , [10/81, 1/81, -8/81] , [1/81, -8/81, 10/81] , [-8/81, 10/81, 1/81] , [1/81, -8/81, 10/81] , [10/81, 1/81, -8/81] ] \$ x \$ [ [9, 0, 6, 12, 0, 0] , [6, 0, 12, 9, 0, 0] , [12, 0, 9, 6, 0, 0] ] \$

Omega Rank for B : cycles: {{3, 5}}, net cycles: 0 . order: 4

$$[y_2, y_1, y_3, 0, y_4, y_5]$$

B = \$ [ [0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] , [0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 1/6, -11/135, -13/270] , [0, 0, 0, 16/135, -11/135] , [0, 0, 0, 16/135, -11/135] , [1/6, -1/12, -1/8, 1/540, 83/1080] , [0, 0, 0, -11/135, 16/135] , [0, 1/6, -1/12, -13/270, 1/540] ] \$ x \$ [ [3, 6, 4, 0, 8, 6] , [6, 3, 8, 0, 10, 0] , [0, 6, 10, 0, 11, 0] , [0, 0, 11, 0, 16, 0] , [0, 0, 16, 0, 11, 0] ] \$

Â» SYNC'D 13/64 , 0.2031250000

31 . Coloring, {3, 4, 5, 6}

**R:** [4, 5, 5, 3, 1, 4] **B:** [2, 4, 1, 6, 3, 1]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [-54' ( ' 3 + τ <sup>2</sup> ' )' , 27' ( ' 3 + τ <sup>2</sup> ' )'' ( ' - 1 + τ ' )' , -27' ( ' 1 + τ ' )'' ( ' 5 - 2τ + τ <sup>2</sup> ' )' , -54' ( ' 3 + τ <sup>2</sup> ' )' , -108' ( ' 1 + τ ' )' , 27' ( ' 3 + τ <sup>2</sup> ' )'' ( ' - 1 + τ ' )' ]'

For  $\tau=1/2$ ,  $[-52, -13, -51, -52, -48, -13]$  . FixedPtCheck,  $[52, 13, 51, 52, 48, 13]$

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ' )'' (' - 1 + \tau ' )' 4$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4, 0], [3/4, 0, 0, 0, 1/4, 0], [0, 0, 1/4, 0, 0, 3/4], [1/4, 0, 3/4, 0, 0, 0], \\ & [3/4, 0, 0, 1/4, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [ [-261123/1236391, 503856/1236391, 10775656/33382557, -6916736/33382557, \\ & 14146816/33382557, -23323136/33382557], [571041/1236391, -11360612/11127519, \\ & -7435880/33382557, -7898816/33382557, -50293760/33382557, 85528576/33382557], \\ & [-1470851/1236391, 11206760/3709173, 24397264/33382557, 49692544/33382557, 82212736/33382557, \\ & -216214016/33382557], [-1439805/1236391, 8500454/3709173, 30442240/33382557, \\ & 44933632/33382557, 88229632/33382557, -199998464/33382557], [3317127/1236391, \\ & -6550386/1236391, -57933296/33382557, -84361376/33382557, -167991296/33382557, \\ & 398820352/33382557], [3814583/3709173, -26140688/11127519, -38417624/33382557, \\ & -38474048/33382557, -67491968/33382557, 189710848/33382557] ] \$ x \$ [ [7, 9/2, 9/2, 9/2, 2, 9/2], \\ & [29/4, 21/4, 21/8, 25/4, 9/4, 27/8], [81/16, 87/16, 13/4, 211/32, 63/32, 75/16], [825/128, 243/64, 25/8, \\ & 417/64, 139/64, 633/128], [3377/512, 2475/512, 417/128, 729/128, 443/256, 1251/256], [3349/512, \\ & 10131/2048, 2787/1024, 1663/256, 4143/2048, 2187/512] ] \$ \end{aligned}$$

Check x AllOnes:  $[1, 1, 1, 1, 1, 1]$

Omega Rank for R : cycles:  $\{\{1, 3, 4, 5\}\}$ , net cycles: 1 . order: 4

$$[y_1, 0, y_2, y_3, y_4, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \$ = \\ & \$ [ [137/945, -214/945, -52/945, 164/945], [-52/945, 164/945, 137/945, -214/945], [-52/945, \\ & 164/945, 137/945, -214/945], [164/945, 137/945, -214/945, -52/945], [-214/945, -52/945, 164/945, \\ & 137/945], [137/945, -214/945, -52/945, 164/945] ] \$ x \$ [ [4, 0, 6, 9, 8, 0], [8, 0, 9, 4, 6, 0], [6, 0, 4, 8, 9, 0], [9, 0, 8, 6, 4, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles:  $\{\{1, 2, 4, 6\}\}$ , net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0, y_5]$$

$$\begin{aligned} B = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, 112/5535, 598/5535, -563/5535, 58/5535], [0, 58/5535, 112/5535, 598/5535, -563/5535], [0, 598/5535, -563/5535, 58/5535, 112/5535], [0, -563/5535, 58/5535, 112/5535, 598/5535], [1/4, -563/5535, 58/5535, 112/5535, -3143/22140], [0, 598/5535, -563/5535, 58/5535, 112/5535] ] \$ x \$ [ \end{aligned}$$

[8, 6, 4, 3, 0, 6] , [10, 8, 0, 6, 0, 3] , [3, 10, 0, 8, 0, 6] , [6, 3, 0, 10, 0, 8] , [8, 6, 0, 3, 0, 10] ] \$

Â» SYNC'D 321/2048 , 0.1567382812

32 . Coloring, {2, 3, 4, 5, 6}

**R:** [4, 4, 5, 3, 1, 4]   **B:** [2, 5, 1, 6, 3, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

' [ '-54' ( ' 3 + τ <sup>2</sup> ' ) ' <sup>2</sup> , 27' ( ' 3 + τ <sup>2</sup> ' ) ' <sup>2</sup> ( ' - 1 + τ ' ) ' , 27' ( ' - 15 - 11τ - 14τ <sup>2</sup> + 10τ <sup>3</sup> - 3τ <sup>4</sup> + τ <sup>5</sup> ' ) ' , 54' ( ' 1 + τ ' ) ' ( ' 3 + τ <sup>2</sup> ' ) ' ( ' - 3 + τ ' ) ' , 108' ( ' - 3 - τ - 5τ <sup>2</sup> + τ <sup>3</sup> ' ) ' , -27' ( ' 1 + τ ' ) ' ( ' 3 + τ <sup>2</sup> ' ) ' ( ' - 1 + τ ' ) ' ( ' - 3 + τ ' ) ' ]'

For τ=1/2, [-676, -169, -733, -780, -592, -195] . FixedPtCheck, [676, 169, 733, 780, 592, 195]

det(A + τ Δ) = 1' ( ' 1 + τ ' ) ' ( ' - 1 + τ ' ) ' <sup>4</sup>

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

\$ [ [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 0, 1/4, 3/4, 0] , [3/4, 0, 0, 0, 1/4, 0] , [0, 0, 1/4, 0, 0, 3/4] , [1/4, 0, 3/4, 0, 0, 0] , [3/4, 0, 0, 1/4, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ =

\$ [ [-20088/546823, 326477/4921407, 3197944/14764221, -5275760/14764221, 419392/14764221, 1768192/14764221] , [-173839/1640469, -1524146/44292663, 1464452/14764221, 1502032/4921407, -5006848/14764221, 4967168/44292663] , [770/14779, -54763/399033, -181568/399033, -39440/399033, -1856/399033, 271616/399033] , [153118/546823, -997615/14764221, 2535820/14764221, 1594240/14764221, -3430208/14764221, -3289600/14764221] , [-57507/546823, -621160/4921407, -582704/14764221, 2145568/14764221, 7979392/14764221, -5579264/14764221] , [10060/1640469, 19218325/44292663, -958348/14764221, 809440/4921407, 503744/14764221, -23770624/44292663] ] \$ x \$ [ [7, 9/2, 9/2, 3, 7/2, 9/2] , [61/8, 21/4, 27/8, 4, 9/2, 9/4] , [171/32, 183/32, 35/8, 121/32, 153/32, 3] , [861/128, 513/128, 145/32, 225/64, 689/128, 363/128] , [1759/256, 2583/512, 2517/512, 1737/512, 2119/512, 675/256] , [1715/256, 5277/1024, 4047/1024, 7451/2048, 5133/1024, 5211/2048] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4, 5}}, net cycles: 1 . order: 4

$$[y_1, 0, y_2, y_3, y_4, 0]$$

$$R = \$ [ [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [1112/10017, -454/10017, -211/10017, -76/10017], [1112/10017, -454/10017, -211/10017, -76/10017], [-211/10017, -76/10017, 1112/10017, -454/10017], [-76/10017, 1112/10017, -454/10017, -211/10017], [-454/10017, -211/10017, -76/10017, 1112/10017], [1112/10017, -454/10017, -211/10017, -76/10017] ] \$ \times \$ [ [4, 0, 6, 12, 5, 0], [5, 0, 12, 4, 6, 0], [6, 0, 4, 5, 12, 0], [12, 0, 5, 6, 4, 0] ] \$$$

Omega Rank for B : cycles: {{1, 2, 3, 5}}, net cycles: 0 . order: 4

$$[y_2, y_1, y_3, 0, y_4, y_5]$$

$$B = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 112/5535, 598/5535, -563/5535, 58/5535], [0, 58/5535, 112/5535, 598/5535, -563/5535], [0, 598/5535, -563/5535, 58/5535, 112/5535], [1/6, -563/5535, 58/5535, 112/5535, -649/11070], [0, -563/5535, 58/5535, 112/5535, 598/5535], [0, 598/5535, -563/5535, 58/5535, 112/5535] ] \$ \times \$ [ [8, 6, 4, 0, 3, 6], [10, 8, 3, 0, 6, 0], [3, 10, 6, 0, 8, 0], [6, 3, 8, 0, 10, 0], [8, 6, 10, 0, 3, 0] ] \$$$

Â» SYNC'D 219/1024 , 0.2138671875

SUMMARY	
<b>Graph Type</b>	NOT CC
$v(A)$	0
$v(\Delta)$	1
$\pi$	[6, 3, 5, 6, 4, 3]
<b>Dbly Stoch</b>	false

SANDWICH		Total 1
No .	Coloring	Rank
<b>1</b>	{4, 5, 6}	3

<b>RT GROUPS</b>		Total 0	
<b>No .</b>	<b>Coloring</b>	<b>Rank</b>	<b>Solv</b>

<b><math>\Delta</math>-RANK'D</b>	<b>SC'D !RK'D</b>	<b><math>\tau</math>-RANK'D</b>	<b>R/B RANK'D</b>	<b>NOT SYNC'D</b>	<b>Total Runs</b>	<b><math>2^{n-1}</math></b>
31	0	31 , 31	27 , 24	1	32	32

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