

New Graph

[2, 1, 5, 5, 1], [4, 3, 4, 3, 2]

$$\pi = [1, 1, 1, 1, 1]$$

POSSIBLE RANKS

1 x 5

BASE DETERMINANT 725/4096, .1770019531

NullSpace of Δ

{1, 2, 3, 4, 5}

Range of Δ : $[-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4]$

1. Coloring, {}

R: [2, 1, 5, 5, 1] **B**: [4, 3, 4, 3, 2]

' See graph

' ' See pair graph

'

Ω for $A+\tau\Delta$:

$['5('1 + \tau')^2 ('-3 + \tau')^2, -5('3 + \tau^2')('1 + \tau')(' - 3 + \tau'), -5(' - 1 + \tau')('9 - \tau - \tau^2 + \tau^3'), 5(' - 1 + \tau')(' - 9 - \tau + \tau^2 + \tau^3'), 5('3 + \tau')(' - 1 + \tau')('1 + \tau')(' - 3 + \tau')']$

For $\tau=1/2$, [225, 195, 67, 73, 105] . FixedPtCheck, [225, 195, 67, 73, 105]

$$\det(A + \tau \Delta) = 1 (' - 1 + \tau')^2 ('1 + \tau')^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	3 vs 3

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [1/4, 0, 3/4, 0, 0], [0, 0, 0, 3/4, 1/4], [0, 0, 3/4, 0, 1/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[2367/1925, 1194/1925, 6808/1925, -2432/1925, -7552/1925], [-573/1925, 1614/1925, \\ & -6352/1925, -192/1925, 5888/1925], [-513/1925, 34/1925, 4064/5775, -2752/1925, 6784/5775], \\ & [-163/1925, 1084/1925, 5464/5775, 2848/1925, -15616/5775], [807/1925, -3926/1925, -3632/1925, \\ & 2528/1925, 4608/1925]] \$ \times \$ [[1/2, 1, 3/2, 3/2, 1/2], [3/8, 1/2, 15/8, 3/2, 3/4], [5/16, 21/32, 3/2, 27/16, \\ & 27/32], [3/8, 91/128, 225/128, 87/64, 51/64], [193/512, 177/256, 795/512, 819/512, 399/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2}}, net cycles: 0 . order: 2

$$[y_1, y_2, 0, 0, y_3]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, -2/5, 3/5], [0, 3/5, -2/5], [1/2, -2/5, \\ & 1/10], [1/2, -2/5, 1/10], [0, 3/5, -2/5]] \$ \times \$ [[2, 1, 0, 0, 2], [3, 2, 0, 0, 0], [2, 3, 0, 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 2

$$[0, y_1, y_2, y_3, 0]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0]] \$ = \$ [[0, -2/5, 3/5], [0, 3/5, -2/5], [0, -2/5, 3/5], \\ & [0, 3/5, -2/5], [1, -2/5, -2/5]] \$ \times \$ [[0, 1, 2, 2, 0], [0, 0, 3, 2, 0], [0, 0, 2, 3, 0]] \$ \end{aligned}$$

Â» SYNC'D 3/16 , 0.1875000000

2. Coloring, {2}

R: [2, 3, 5, 5, 1] **B:** [4, 1, 4, 3, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$['5' ('1 + \tau') ' (' - 3 + \tau') ' , 5' ('1 + \tau') ' (' - 3 + \tau') ' , -5' ('3 + \tau^2') ' , 5' (' - 1 + \tau') ' ('3 + \tau') ' , 5' ('1 + \tau') ' (' - 3 + \tau') ']'$$

For τ=1/2, [-15, -15, -13, -7, -15] . FixedPtCheck, [15, 15, 13, 7, 15]

$$\det(A + \tau \Delta) = 1' (' - 1 + \tau') ' ('1 + 3\tau^2') ' ('1 + \tau')'$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\begin{aligned}
& \$ [[0, 1/4, 0, 3/4, 0], [3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4], [0, 0, 3/4, 0, 1/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\
& 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\
& \$ [[3/4, -7/20, 19/5, -4/5, -16/5], [-27/52, 599/260, -283/65, -92/65, 272/65], [-57/65, 14/65, -8/13, \\
& -32/13, 256/65], [-23/130, -11/130, 38/65, 152/65, -32/13], [237/130, -271/130, 38/65, 152/65, -32/13]] \\
& \$ \times \$ [[1, 1, 1, 3/2, 1/2], [7/8, 5/8, 11/8, 3/2, 5/8], [5/8, 11/16, 41/32, 27/16, 23/32], [89/128, 89/128, \\
& 23/16, 183/128, 95/128], [181/256, 187/256, 319/256, 819/512, 367/512]] \$
\end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2, 3, 5}}, net cycles: 1 . order: 4

$$[y_1, y_2, y_3, 0, y_4]$$

$$\begin{aligned}
R = & \$ [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\
& 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[-1/5, -1/5, 4/5, -1/5], [-1/5, -1/5, -1/5, 4/5] \\
& , [4/5, -1/5, -1/5, -1/5], [4/5, -1/5, -1/5, -1/5], [-1/5, 4/5, -1/5, -1/5]] \$ \times \$ [[1, 1, 1, 0, 2], [2, 1, 1, 0, 1], \\
& [1, 2, 1, 0, 1], [1, 1, 2, 0, 1]] \$
\end{aligned}$$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0]$$

$$\begin{aligned}
B = & \$ [[0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\
& 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0]] \$ = \$ [[0, 0, 3/5, -2/5], [0, 1, -2/5, -2/5], [0, 0, \\
& 3/5, -2/5], [0, 0, -2/5, 3/5], [1, -1, -2/5, 3/5]] \$ \times \$ [[1, 1, 1, 2, 0], [1, 0, 2, 2, 0], [0, 0, 2, 3, 0], [0, 0, 3, \\
& 2, 0]] \$
\end{aligned}$$

\hat{A} » SYNC'D 9/64 , 0.1406250000

3 . Coloring, {3}

R: [2, 1, 4, 5, 1] **B**: [4, 3, 5, 3, 2]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$\begin{aligned}
& ['5' ('1 + \tau') ' ' ('3 + \tau^2') ' ' (' - 3 + \tau') ' , -5' ('3 + \tau^2') ' ^2 , 5' ('9 - \tau - \tau^2 + \tau^3') ' ' (' - 1 + \\
& \tau') ' , 5' ('9 - 2\tau + \tau^2') ' ' ('1 + \tau') ' ' (' - 1 + \tau') ' , 5' ('3 + \tau') ' ' ('3 + \tau^2') ' ' (' - 1 + \tau') ' '] '
\end{aligned}$$

For $\tau=1/2$, $[-195, -169, -67, -99, -91]$. FixedPtCheck, $[195, 169, 67, 99, 91]$

$$\det(A + \tau \Delta) = 1 \cdot (1 + \tau) \cdot (1 + 3\tau^2) \cdot (-1 + \tau)$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [1/4, 0, 3/4, 0, 0], [0, 0, 0, 1/4, 3/4], [0, 0, 3/4, 0, 1/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[4611/2885, -5778/2885, 2352/2885, -6624/2885, 6016/2885], [-2139/2885, 732/2885, \\ & -7488/2885, 6016/2885, 3456/2885], [-1159/2885, -8/2885, 2352/2885, -6624/2885, 6016/2885], \\ & [1371/2885, 7502/2885, -1448/2885, 3136/2885, -9984/2885], [201/2885, -2448/2885, 4232/2885, \\ & 4096/2885, -5504/2885]] \$ \times \$ [[1/2, 1, 3/2, 1, 1], [1/2, 7/8, 3/2, 3/4, 11/8], [9/16, 37/32, 39/32, 3/4, \\ & 21/16], [79/128, 9/8, 183/128, 93/128, 141/128], [285/512, 251/256, 711/512, 105/128, 321/256]] \$ \end{aligned}$$

Check x AllOnes: $[1, 1, 1, 1, 1]$

Omega Rank for R : cycles: $\{\{1, 2\}\}$, net cycles: 0 . order: 4

$$[y_1, y_2, 0, y_3, y_4]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [1, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 0, -2/5, 3/5], [0, 0, 3/5, -2/5], [1, -1, \\ & -2/5, 3/5], [0, 1, -2/5, -2/5], [0, 0, 3/5, -2/5]] \$ \times \$ [[2, 1, 0, 1, 1], [2, 2, 0, 0, 1], [3, 2, 0, 0, 0], [2, 3, 0, \\ & 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: $\{\{2, 3, 5\}\}$, net cycles: 0 . order: 3

$$[0, y_1, y_2, y_3, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[1, -3/5, 2/5, -3/5], [0, 2/5, -3/5, 2/5], [0, \\ & 2/5, 2/5, -3/5], [0, 2/5, -3/5, 2/5], [0, -3/5, 2/5, 2/5]] \$ \times \$ [[0, 1, 2, 1, 1], [0, 1, 2, 0, 2], [0, 2, 1, 0, 2], \\ & [0, 2, 2, 0, 1]] \$ \end{aligned}$$

Â» SYNC'D 31/256 , 0.1210937500

4 . Coloring, $\{4\}$

R: $[2, 1, 5, 3, 1]$ **B:** $[4, 3, 4, 5, 2]$

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$[-5(1+\tau)^2(3+\tau^2)(-3+\tau), 5(3+\tau^2)^2, 5(-1+\tau)(-9-5\tau-3\tau^2+\tau^3), 5(-1+\tau)(-9-\tau+\tau^2+\tau^3), -5(-1+\tau)(3+\tau)(3+\tau^2)]$$

For $\tau=1/2$, [195, 169, 97, 73, 91] . FixedPtCheck, [195, 169, 97, 73, 91]

$$\det(A + \tau \Delta) = (-1 + \tau)^2 (1 + \tau)^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [1/4, 0, 3/4, 0, 0], [0, 0, 0, 3/4, 1/4], [0, 0, 1/4, 0, 3/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[3753/1705, 12/31, 984/1705, -1728/1705, -3328/1705], [-1497/1705, -54/31, -4056/1705, 5792/1705, 3072/1705], [-1107/1705, -8/31, -3136/1705, -3808/1705, 8832/1705], [343/1705, 74/31, 984/1705, -1728/1705, -3328/1705], [213/1705, -24/31, 5224/1705, 1472/1705, -5248/1705]] \$ \times \$ [[1/2, 1, 1, 3/2, 1], [1/2, 7/8, 9/8, 9/8, 11/8], [9/16, 37/32, 15/16, 39/32, 9/8], [73/128, 63/64, 75/64, 9/8, 147/128], [273/512, 257/256, 261/256, 669/512, 291/256]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, 0, y_4]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[0, 0, -2/5, 3/5], [0, 0, 3/5, -2/5], [0, 1, -2/5, -2/5], [1, -1, -2/5, 3/5], [0, 0, 3/5, -2/5]] \$ \times \$ [[2, 1, 1, 0, 1], [2, 2, 0, 0, 1], [3, 2, 0, 0, 0], [2, 3, 0, 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 3, 4, 5}}, net cycles: 1 . order: 4

$$[0, y_4, y_1, y_2, y_3]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[4/5, -1/5, -1/5, -1/5], [-1/5, -1/5, -1/5, 4/5], [4/5, -1/5, -1/5, -1/5], [-1/5, 4/5, -1/5, -1/5], [-1/5, -1/5, 4/5, -1/5]] \$ \times \$ [[0, 1, 1, 2, 1], [0, 1, 1, 1, 2], [0, 2, 1, 1, 1], [0, 1, 2, 1, 1]] \$ \end{aligned}$$

Â» SYNC'D 11/64 , 0.1718750000

5 . Coloring, {5}

R: [2, 1, 5, 5, 2] **B:** [4, 3, 4, 3, 1]

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

' ['-5' (' 1 + τ ')'' (' 3 + τ^2 ')'' (' - 3 + τ ')', 5' (' 1 + τ ')'^2 (' - 3 + τ ')'^2 , 5' (' - 1 + τ ')'' (' - 9 - τ + τ^2 + τ^3 ')', -5' (' - 1 + τ ')'' (' 9 - τ - τ^2 + τ^3 ')', 5' (' 1 + τ ')'' (' - 1 + τ ')'' (' 3 + τ ')'' (' - 3 + τ ')'']'

For $\tau=1/2$, [195, 225, 73, 67, 105] . FixedPtCheck, [195, 225, 73, 67, 105]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')'^2 (' - 1 + \tau ')'^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	3 vs 3

bi =

\$ [[0, 1/4, 0, 3/4, 0] , [1/4, 0, 3/4, 0, 0] , [0, 0, 0, 3/4, 1/4] , [0, 0, 3/4, 0, 1/4] , [3/4, 1/4, 0, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0] , [0, 1, 0, 0, 0] , [0, 0, 1, 0, 0] , [0, 0, 0, 1, 0] , [0, 0, 0, 0, 1]] \$ =
 \$ [[-573/1925, 1614/1925, -6352/1925, -192/1925, 5888/1925] , [2367/1925, 1194/1925, 6808/1925, -2432/1925, -7552/1925] , [-163/1925, 1084/1925, 5464/5775, 2848/1925, -15616/5775] , [-513/1925, 34/1925, 4064/5775, -2752/1925, 6784/5775] , [807/1925, -3926/1925, -3632/1925, 2528/1925, 4608/1925]] \$ x \$ [[1, 1/2, 3/2, 3/2, 1/2] , [1/2, 3/8, 3/2, 15/8, 3/4] , [21/32, 5/16, 27/16, 3/2, 27/32] , [91/128, 3/8, 87/64, 225/128, 51/64] , [177/256, 193/512, 819/512, 795/512, 399/512]] \$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2}}, net cycles: 0 . order: 2

$$[y_1, y_2, 0, 0, y_3]$$

R = \$ [[0, 1, 0, 0, 0] , [1, 0, 0, 0, 0] , [0, 0, 0, 0, 1] , [0, 0, 0, 0, 1] , [0, 1, 0, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0] , [0, 1, 0, 0, 0] , [0, 0, 0, 0, 0] , [0, 0, 0, 0, 1]] \$ = \$ [[0, 3/5, -2/5] , [0, -2/5, 3/5] , [1/2, -2/5, 1/10] , [1/2, -2/5, 1/10] , [0, 3/5, -2/5]] \$ x \$ [[1, 2, 0, 0, 2] , [2, 3, 0, 0, 0] , [3, 2, 0, 0, 0]] \$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 2

$$[y_1, 0, y_2, y_3, 0]$$

$$B = \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0]] \$ = \$ [[0, 3/5, -2/5], [0, -2/5, 3/5], [0, 3/5, -2/5], [0, -2/5, 3/5], [1, -2/5, -2/5]] \$ \times \$ [[1, 0, 2, 2, 0], [0, 0, 2, 3, 0], [0, 0, 3, 2, 0]] \$$$

Â» SYNC'D 3/16 , 0.1875000000

6 . Coloring, {2, 3}

R: [2, 3, 4, 5, 1] **B:** [4, 1, 5, 3, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[' 1 , 1 , 1 , 1 , 1 ']$$

For τ=1/2, [1, 1, 1, 1, 1] . FixedPtCheck, [1, 1, 1, 1, 1]

$$\det(A + \tau \Delta) = 1 (1 + 10\tau^2 + 5\tau^4)$$

Delta Range : [-y₁ - y₂ - y₃ - y₄, y₁, y₂, y₃, y₄]

$$[1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; - \quad \backslash ; \Delta$$

$$\$ [[1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1]] \$ \quad \$ [[1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1]] \$ \quad \$ [[0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0]] \$$$

$$[0, 0, 0, 0, 0]$$

$$p = s$$

$$S+ \quad \backslash ; S- \quad \backslash ; NM$$

$$\$ [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \quad \$ [[0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0]] \$ \quad \$ [[4, 3, 3, 3, 3], [3, 4, 3, 3, 3], [3, 3, 4, 3, 3], [3, 3, 3, 4, 3], [3, 3, 3, 3, 4]] \$$$

CmmCk true, true, true

$$p' = s \quad p' = s^2 \quad p' = s^3$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
0 vs 4	1 vs 5	1 vs 5	1 vs 5	1 vs 5

Omega Rank for R : cycles: $\{\{1, 2, 3, 4, 5\}\}$, net cycles: 1 . order: 5

$$\$ [[1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1]] \$$$

$$[y_1, y_1, y_1, y_1, y_1]$$

$$p' = -1 + s \quad p' = -1 + s^2 \quad p' = -1 + s^3 \quad p' = -1 + s^4$$

Omega Rank for B : cycles: $\{\{1, 2, 3, 4, 5\}\}$, net cycles: 1 . order: 5

$$\$ [[1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1]] \$$$

$$[y_1, y_1, y_1, y_1, y_1]$$

$$p' = -1 + s \quad p' = -1 + s^2 \quad p' = -1 + s^3 \quad p' = -1 + s^4$$

Â« NOT SYNC'D Â»

Nullspace of $\{\Omega\Delta^i\}$:

$$[x_4, x_3, x_2, x_1]$$

$$\text{For } A+2\Delta : [y_3, y_4, y_2, -y_1 - y_2 - y_3 - y_4, y_1]$$

$$\text{For } A-2\Delta : [y_1, -y_1 - y_2 - y_3 - y_4, y_2, y_3, y_4]$$

Range of $\{\Omega\Delta^i\}$: $[0, 0, 0, 0, 0]$

rank of M is 5 , rank of N is 5

M N

$$\$ [[0, 1, 1, 1, 1], [1, 0, 1, 1, 1], [1, 1, 0, 1, 1], [1, 1, 1, 0, 1], [1, 1, 1, 1, 0]] \$ \quad \$ [[0, 1, 1, 1, 1], [1, 0, 1, 1, 1], [1, 1, 0, 1, 1], [1, 1, 1, 0, 1], [1, 1, 1, 1, 0]] \$$$

Check is $\Omega\Delta N$ zero? *true*, $\pi\Delta = [0, 0, 0, 0, 0]$

ker M, $[0, 0, 0, 0, 0]$

Range M, $[x_1, x_2, x_3, x_4, x_5]$

$$\tau = 5, r' = 4/5$$

Ranges

Action of R on ranges, [[1]]

Action of B on ranges, [[1]]

$$\beta(\{1, 2, 3, 4, 5\}) = 1/1$$

ker N, [0, 0, 0, 0, 0]

Range of N

$$[y_1, y_2, y_3, y_4, y_5]$$

Partitions

$$\alpha(\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}) = 1/1$$

$$b_1 = \{1\} \text{ , ' b}_2 = \{2\} \text{ , ' b}_3 = \{3\} \text{ , ' b}_4 = \{4\} \text{ , ' b}_5 = \{5\}$$

Action of R and B on the blocks of the partitions: \$ [[0, 1, 0, 0, 1] , [1, 0, 0, 0, 1] , [0, 1, 0, 1, 0] , [1, 0, 1, 0, 0] , [0, 0, 1, 1, 0]] \$ = \$ [[0, 0, 0, 0, 1] , [1, 0, 0, 0, 0] , [0, 1, 0, 0, 0] , [0, 0, 1, 0, 0] , [0, 0, 0, 1, 0]] \$ + \$ [[0, 1, 0, 0, 0] , [0, 0, 0, 0, 1] , [0, 0, 0, 1, 0] , [1, 0, 0, 0, 0] , [0, 0, 1, 0, 0]] \$
[‘5‘ , ‘1‘ , ‘2‘ , ‘3‘ , ‘4‘], [‘2‘ , ‘5‘ , ‘4‘ , ‘1‘ , ‘3‘] with invariant measure [1, 1, 1, 1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-5 partition graph.

‘

Right Group	
Coloring	{2, 3}
Rank	5
R,B	[2, 3, 4, 5, 1], [4, 1, 5, 3, 2]
π_2	[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
u_2	[1, 1, 1, 1, 1, 1, 1, 1, 1, 1] (dim 1)
wpp	[1, 1, 1, 1, 1]
π_5	[1]
u_5	[1]

7 . Coloring, {2, 4}

$$\mathbf{R}: [2, 3, 5, 3, 1] \quad \mathbf{B}: [4, 1, 4, 5, 2]$$

' See graph

' ' See pair graph

Ω for $A+\tau\Delta$:

$$['5' ('3 + \tau^2 ') ' , 5' ('3 + \tau^2 ') ' , -5' ('1 + \tau ') ' (' -3 + \tau ') ' , -5' ('3 + \tau ') ' (' -1 + \tau ') ' , 5' ('3 + \tau^2 ') ']'$$

For $\tau=1/2$, [13, 13, 15, 7, 13] . FixedPtCheck, [13, 13, 15, 7, 13]

$$\det(A + \tau \Delta) = 1' (' -1 + \tau ') ' ^2 ('1 + \tau ') ' ^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4], [0, 0, 1/4, 0, 3/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-9/10, -17/10, -2, -8/5, 32/5], [33/130, -131/130, -2/13, 296/65, -224/65], [144/65, 37/65, -8/13, \\ & -64/65, -64/65], [14/65, 167/65, -8/13, -64/65, -64/65], [-51/65, -28/65, 44/13, -64/65, -64/65]] \$ \times \$ [\\ & [1, 1, 1/2, 3/2, 1], [1, 1, 5/8, 9/8, 5/4], [17/16, 19/16, 17/32, 39/32, 1], [73/64, 65/64, 77/128, 153/128, \\ & 67/64], [131/128, 137/128, 283/512, 669/512, 67/64]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2, 3, 5}}, net cycles: 1 . order: 4

$$[y_1, y_4, y_2, 0, y_3]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[-1/5, -1/5, -1/5, 4/5], [4/5, -1/5, -1/5, -1/5] \\ & , [-1/5, 4/5, -1/5, -1/5], [4/5, -1/5, -1/5, -1/5], [-1/5, -1/5, 4/5, -1/5]] \$ \times \$ [[1, 1, 2, 0, 1], [1, 1, 1, 0, 2], \\ & [2, 1, 1, 0, 1], [1, 2, 1, 0, 1]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 4, 5}}, net cycles: 1 . order: 4

$$[y_3, y_4, 0, y_1, y_2]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[4/5, -1/5, -1/5, -1/5], [-1/5, -1/5, -1/5, 4/5] \\ & , [4/5, -1/5, -1/5, -1/5], [-1/5, 4/5, -1/5, -1/5], [-1/5, -1/5, 4/5, -1/5]] \$ \times \$ [[1, 1, 0, 2, 1], [1, 1, 0, 1, 2], \\ & [1, 2, 0, 1, 1], [2, 1, 0, 1, 1]] \$ \end{aligned}$$

Â» SYNC'D 27/128 , 0.2109375000

8. Coloring, {2, 5}

R: [2, 3, 5, 5, 2] **B:** [4, 1, 4, 3, 1]

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

' ['5' (' 1 + \tau ')'' (' - 1 + \tau ')'' (' 3 + \tau ')'' (' - 3 + \tau ')' , 5' (' 1 + \tau ')' ^ 2 ' (' - 3 + \tau ')' ^ 2 , -5' (' - 9 - 2\tau - 8\tau ^ 2 + 2\tau ^ 3 + \tau ^ 4 ')' , 5' (' - 1 + \tau ')'' (' - 9 - \tau + \tau ^ 2 + \tau ^ 3 ')' , -5' (' 1 + \tau ')'' (' 3 + \tau ^ 2 ')'' (' - 3 + \tau ')'']'

For $\tau=1/2$, [105, 225, 187, 73, 195] . FixedPtCheck, [105, 225, 187, 73, 195]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')' ^ 2 ' (' - 1 + \tau ')' ^ 2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	3 vs 3

bi =

\$ [[0, 1/4, 0, 3/4, 0] , [3/4, 0, 1/4, 0, 0] , [0, 0, 0, 3/4, 1/4] , [0, 0, 3/4, 0, 1/4] , [3/4, 1/4, 0, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0] , [0, 1, 0, 0, 0] , [0, 0, 1, 0, 0] , [0, 0, 0, 1, 0] , [0, 0, 0, 0, 1]] \$ =
 \$ [[-33/290, 973/870, -178/145, 8/145, 32/87] , [297/290, -19/290, 442/145, -72/145, -96/29] ,
 [-18/145, -1/29, 196/145, 304/145, -448/145] , [-47/145, -44/435, 16/29, -32/29, 512/435] , [78/145,
 -133/145, -108/29, -16/29, 704/145]] \$ x \$ [[3/2, 1/2, 1, 3/2, 1/2] , [3/4, 1/2, 5/4, 15/8, 5/8] , [27/32,
 11/32, 49/32, 3/2, 25/32] , [27/32, 13/32, 155/128, 57/32, 97/128] , [447/512, 205/512, 23/16, 789/512,
 383/512]] \$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 3, 5}}, net cycles: 1 . order: 3

$$[0, y_2, y_1, 0, y_3]$$

$R = $ [[0, 1, 0, 0, 0] , [0, 0, 1, 0, 0] , [0, 0, 0, 0, 1] , [0, 0, 0, 0, 1] , [0, 1, 0, 0, 0]] $ x $ [[0, 0, 0, 0, 0] , [0, 1, 0, 0, 0] , [0, 0, 1, 0, 0] , [0, 0, 0, 0, 0] , [0, 0, 0, 0, 1]] $ = $ [[2/5, 2/5, -3/5] , [-3/5, 2/5, 2/5] , [2/5, -3/5, 2/5] , [2/5, -3/5, 2/5] , [2/5, 2/5, -3/5]] $ x $ [[0, 2, 1, 0, 2] , [0, 2, 2, 0, 1] , [0, 1, 2, 0, 2]] $$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 2

$$[y_1, 0, y_2, y_3, 0]$$

$$B = \$ [[0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0]] \$ = \$ [[0, 3/5, -2/5], [1/2, -2/5, 1/10], [0, 3/5, -2/5], [0, -2/5, 3/5], [1/2, -2/5, 1/10]] \$ \times \$ [[2, 0, 1, 2, 0], [0, 0, 2, 3, 0], [0, 0, 3, 2, 0]] \$$$

Â» SYNC'D 11/32 , 0.3437500000

9 . Coloring, {3, 4}

R: [2, 1, 4, 3, 1] **B:** [4, 3, 5, 5, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[' 5' (' 1 + \tau ')'' (' 3 + \tau ')'' (' - 3 + \tau ')' , -5' (' 3 + \tau ')'' (' 3 + \tau ^ 2 ')' , 5' (' - 9 - 5\tau - 3\tau ^ 2 + \tau ^ 3 ')' , -5' (' 1 + \tau ')'' (' 9 - 2\tau + \tau ^ 2 ')' , 5' (' 3 + \tau ')' ^ 2 (' - 1 + \tau ')'']'$$

For τ=1/2, [-105, -91, -97, -99, -49] . FixedPtCheck, [105, 91, 97, 99, 49]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')' ^ 2 (' - 1 + \tau ')' ^ 2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	2 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [1/4, 0, 3/4, 0, 0], [0, 0, 0, 1/4, 3/4], [0, 0, 1/4, 0, 3/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[273/125, -152/75, 944/375, -1184/375, 256/375], [-87/125, 88/75, -136/375, 1696/375, \\ & -1664/375], [-117/125, -142/75, -976/375, -64/375, 2176/375], [33/125, 208/75, -776/375, 736/375, \\ & -1024/375], [23/125, -2/75, 944/375, -1184/375, 256/375]] \$ \times \$ [[1/2, 1, 1, 1, 3/2], [5/8, 5/4, 1, 5/8, \\ & 3/2], [11/16, 41/32, 35/32, 23/32, 39/32], [5/8, 139/128, 73/64, 101/128, 87/64], [313/512, 301/256, \\ & 259/256, 193/256, 741/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2}, {3, 4}}, net cycles: 2 . order: 2

$$\$ [[2, 1, 1, 1, 0], [1, 2, 1, 1, 0], [2, 1, 1, 1, 0], [1, 2, 1, 1, 0]] \$$$

$$[-y_1 + 3y_2, y_1, y_2, y_2, 0]$$

$$p = -s + s^3 \quad p' = -s + s^3$$

Omega Rank for B : cycles: {{2, 3, 5}}, net cycles: 0 . order: 3

$$[0, y_1, y_2, y_3, y_4]$$

$$B = \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[1, -3/5, 2/5, -3/5], [0, -3/5, 2/5, 2/5], [0, 2/5, -3/5, 2/5], [0, 2/5, -3/5, 2/5], [0, 2/5, 2/5, -3/5]] \$ \times \$ [[0, 1, 1, 1, 2], [0, 2, 1, 0, 2], [0, 2, 2, 0, 1], [0, 1, 2, 0, 2]] \$$$

Â» SYNC'D 9/256 , 0.03515625000

10 . Coloring, {3, 5}

R: [2, 1, 4, 5, 2] **B:** [4, 3, 5, 3, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$['5' ('3 + \tau^2')^2, -5' ('3 + \tau^2') ('1 + \tau') ('-3 + \tau')^2, 5' ('-1 + \tau') ('-9 - \tau + \tau^2 + \tau^3')^2, 5' ('-1 + \tau') ('-9 - 5\tau - 3\tau^2 + \tau^3')^2, -5' ('3 + \tau') ('3 + \tau^2') ('-1 + \tau')^2]$$

For τ=1/2, [169, 195, 73, 97, 91] . FixedPtCheck, [169, 195, 73, 97, 91]

$$\det(A + \tau \Delta) = 1' ('-1 + \tau')^2 ('1 + \tau')^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\$ [[0, 1/4, 0, 3/4, 0], [1/4, 0, 3/4, 0, 0], [0, 0, 0, 1/4, 3/4], [0, 0, 3/4, 0, 1/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ =$$

$$\$ [[-1497/1705, -54/31, -4056/1705, 5792/1705, 3072/1705], [3753/1705, 12/31, 984/1705, -1728/1705, -3328/1705], [343/1705, 74/31, 984/1705, -1728/1705, -3328/1705], [-1107/1705, -8/31, -3136/1705, -3808/1705, 8832/1705], [213/1705, -24/31, 5224/1705, 1472/1705, -5248/1705]] \$ \times \$ [[1, 1/2, 3/2, 1, 1], [7/8, 1/2, 9/8, 9/8, 11/8], [37/32, 9/16, 39/32, 15/16, 9/8], [63/64, 73/128, 9/8, 75/64, 147/128], [257/256, 273/512, 669/512, 261/256, 291/256]] \$$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2}}, net cycles: 0 . order: 4

$$[y_1, y_3, 0, y_2, y_4]$$

$$R = \$ [[0, 1, 0, 0, 0], [1, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 0, 3/5, -2/5], [0, 0, -2/5, 3/5], [1, -1, -2/5, 3/5], [0, 1, -2/5, -2/5], [0, 0, 3/5, -2/5]] \$ \times \$ [[1, 2, 0, 1, 1], [2, 2, 0, 0, 1], [2, 3, 0, 0, 0], [3, 2, 0, 0, 0]] \$$$

Omega Rank for B : cycles: {{1, 3, 4, 5}}, net cycles: 1 . order: 4

$$[y_4, 0, y_1, y_2, y_3]$$

$$B = \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[-1/5, -1/5, -1/5, 4/5], [4/5, -1/5, -1/5, -1/5], [-1/5, 4/5, -1/5, -1/5], [4/5, -1/5, -1/5, -1/5], [-1/5, -1/5, 4/5, -1/5]] \$ \times \$ [[1, 0, 2, 1, 1], [1, 0, 1, 1, 2], [2, 0, 1, 1, 1], [1, 0, 1, 2, 1]] \$$$

Â» SYNC'D 11/64 , 0.1718750000

11 . Coloring, {4, 5}

R: [2, 1, 5, 3, 2] **B:** [4, 3, 4, 5, 1]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$[-5(3 + \tau^2)^2, 5(3 + \tau^2)(1 + \tau)(-3 + \tau), 5(-1 + \tau)(9 - 2\tau + \tau^2)(1 + \tau), 5(-1 + \tau)(9 - \tau - \tau^2 + \tau^3), 5(-1 + \tau)(3 + \tau)(3 + \tau^2)]$$

For τ=1/2, [-169, -195, -99, -67, -91] . FixedPtCheck, [169, 195, 99, 67, 91]

$$\det(A + \tau \Delta) = 1(-1 + \tau)(1 + 3\tau^2)(1 + \tau)$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [1/4, 0, 3/4, 0, 0], [0, 0, 0, 3/4, 1/4], [0, 0, 1/4, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-2139/2885, 732/2885, -7488/2885, 6016/2885, 3456/2885], [4611/2885, -5778/2885, \\ & 2352/2885, -6624/2885, 6016/2885], [1371/2885, 7502/2885, -1448/2885, 3136/2885, -9984/2885], \\ & [-1159/2885, -8/2885, 2352/2885, -6624/2885, 6016/2885], [201/2885, -2448/2885, 4232/2885, \\ & 4096/2885, -5504/2885]] \$ \times \$ [[1, 1/2, 1, 3/2, 1], [7/8, 1/2, 3/4, 3/2, 11/8], [37/32, 9/16, 3/4, 39/32, \\ & 21/16], [9/8, 79/128, 93/128, 183/128, 141/128], [251/256, 285/512, 105/128, 711/512, 321/256]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, 0, y_4]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 0, 3/5, -2/5], [0, 0, -2/5, 3/5], [0, 1, \\ & -2/5, -2/5], [1, -1, -2/5, 3/5], [0, 0, 3/5, -2/5]] \$ \times \$ [[1, 2, 1, 0, 1], [2, 2, 0, 0, 1], [2, 3, 0, 0, 0], [3, 2, 0, \\ & 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 2/5, -3/5, 2/5], [1, -3/5, 2/5, -3/5], [0, \\ & 2/5, -3/5, 2/5], [0, 2/5, 2/5, -3/5], [0, -3/5, 2/5, 2/5]] \$ \times \$ [[1, 0, 1, 2, 1], [1, 0, 0, 2, 2], [2, 0, 0, 1, 2], \\ & [2, 0, 0, 2, 1]] \$ \end{aligned}$$

Â» SYNC'D 31/256 , 0.1210937500

12 . Coloring, {2, 3, 4}

R: [2, 3, 4, 3, 1] **B:** [4, 1, 5, 5, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$['5' ('3 + \tau')' ('-1 + \tau')', 5' ('3 + \tau')' ('-1 + \tau')', 5' ('-3 + \tau')' ('1 + \tau')', -5' ('3 + \tau^2')', 5' ('3 + \tau')' ('-1 + \tau')']'$$

For τ=1/2, [-7, -7, -15, -13, -7] . FixedPtCheck, [7, 7, 15, 13, 7]

$$\det(A + \tau \Delta) = 1' (' 1 + 3\tau^2 ')'' (' - 1 + \tau ')'' (' 1 + \tau ')'$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4], [0, 0, 1/4, 0, 3/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-57/100, 1/4, -31/25, 92/25, -48/25], [9/1300, -57/52, 847/325, -604/325, 176/325], [1227/650, \\ & 3/26, -18/325, -24/325, -544/325], [-69/325, -18/13, -408/325, -544/325, 1536/325], [-73/650, 55/26, \\ & -18/325, -24/325, -544/325]] \$ \times \$ [[1, 1, 1/2, 1, 3/2], [9/8, 11/8, 1/2, 7/8, 9/8], [21/16, 9/8, 9/16, 31/32, \\ & 33/32], [141/128, 141/128, 67/128, 9/8, 147/128], [285/256, 291/256, 285/512, 245/256, 633/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0]] \$ = \$ [[0, 1, -2/5, -2/5], [0, 0, 3/5, -2/5], [0, 0, \\ & -2/5, 3/5], [0, 0, 3/5, -2/5], [1, -1, -2/5, 3/5]] \$ \times \$ [[1, 1, 2, 1, 0], [0, 1, 2, 2, 0], [0, 0, 3, 2, 0], [0, 0, 2, \\ & 3, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 4, 5}}, net cycles: 1 . order: 4

$$[y_1, y_2, 0, y_3, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[-1/5, -1/5, -1/5, 4/5], [-1/5, -1/5, 4/5, -1/5] \\ & , [4/5, -1/5, -1/5, -1/5], [4/5, -1/5, -1/5, -1/5], [-1/5, 4/5, -1/5, -1/5]] \$ \times \$ [[1, 1, 0, 1, 2], [1, 2, 0, 1, 1], \\ & [2, 1, 0, 1, 1], [1, 1, 0, 2, 1]] \$ \end{aligned}$$

Â» SYNC'D 9/64 , 0.1406250000

13 . Coloring, {2, 3, 5}

R: [2, 3, 4, 5, 2] **B:** [4, 1, 5, 3, 1]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[5(3+\tau)^{-1}(-1+\tau)^{-1}(3+\tau^2)^{-1}, 5(1+\tau)^{-1}(3+\tau^2)^{-1}(-3+\tau)^{-1}, 5(-9-2\tau-8\tau^2+2\tau^3+\tau^4)^{-1}, 5(-9+2\tau-8\tau^2-2\tau^3+\tau^4)^{-1}, -5(3+\tau^2)^{-2}]$$

For τ=1/2, [-91, -195, -187, -163, -169] . FixedPtCheck, [91, 195, 187, 163, 169]

$$\det(A + \tau \Delta) = 1(1+3\tau^2)^{-1}(-1+\tau)^{-1}(1+\tau)^{-1}$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4], [0, 0, 3/4, 0, 1/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-2/25, 61/25, 2/5, -32/25, -32/25], [48/25, 11/25, 2/5, -32/25, -32/25], [-9/50, -73/50, -2/5, \\ & 88/25, -32/25], [-39/50, -43/50, 14/5, 8/25, -32/25], [3/25, -14/25, -16/5, -32/25, 128/25]] \$ \times \$ [[3/2, \\ & 1/2, 1, 1, 1], [9/8, 5/8, 7/8, 11/8, 1], [39/32, 17/32, 19/16, 17/16, 1], [147/128, 71/128, 119/128, 155/128, \\ & 37/32], [657/512, 295/512, 67/64, 35/32, 1]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 3, 4, 5}}, net cycles: 1 . order: 4

$$[0, y_1, y_2, y_3, y_4]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[4/5, -1/5, -1/5, -1/5], [-1/5, 4/5, -1/5, -1/5] \\ & , [-1/5, -1/5, 4/5, -1/5], [-1/5, -1/5, -1/5, 4/5], [4/5, -1/5, -1/5, -1/5]] \$ \times \$ [[0, 2, 1, 1, 1], [0, 1, 2, 1, 1], \\ & [0, 1, 1, 2, 1], [0, 1, 1, 1, 2]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3, 4, 5}}, net cycles: 1 . order: 4

$$[y_1, 0, y_2, y_3, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[-1/5, 4/5, -1/5, -1/5], [4/5, -1/5, -1/5, -1/5] \\ & , [-1/5, -1/5, -1/5, 4/5], [-1/5, -1/5, 4/5, -1/5], [4/5, -1/5, -1/5, -1/5]] \$ \times \$ [[2, 0, 1, 1, 1], [1, 0, 1, 2, 1], \\ & [1, 0, 2, 1, 1], [1, 0, 1, 1, 2]] \$ \end{aligned}$$

Â» SYNC'D 25/128 , 0.1953125000

14 . Coloring, {2, 4, 5}

R: [2, 3, 5, 3, 2] **B:** [4, 1, 4, 5, 1]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$[-5' (' 3 + \tau ')'' (' 3 + \tau ^ 2 ')'' (' - 1 + \tau ')' , -5' (' 1 + \tau ')'' (' 3 + \tau ^ 2 ')'' (' - 3 + \tau ')' , 5' (' 1 + \tau ')'' (' 9 - \tau - \tau ^ 2 + \tau ^ 3 ')' , 5' (' - 1 + \tau ')'' (' - 9 - \tau + \tau ^ 2 + \tau ^ 3 ')' , 5' (' 3 + \tau ^ 2 ')'' ^ 2 ']'$$

For $\tau=1/2$, [91, 195, 201, 73, 169] . FixedPtCheck, [91, 195, 201, 73, 169]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')'' ^ 2 (' - 1 + \tau ')'' ^ 2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	3 vs 3

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [3/4, 0, 1/4, 0, 0], [0, 0, 0, 3/4, 1/4], [0, 0, 1/4, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-17/90, 113/270, -20/9, 56/45, 128/135], [11/30, -359/90, -4/3, -8/15, 256/45], [41/30, 331/90, 2/3, -8/15, -224/45], [-7/90, 343/270, 14/9, -104/45, -32/135], [-7/15, -62/45, 4/3, 32/15, -64/45]] \$ \times \$ \\ & [[3/2, 1/2, 1/2, 3/2, 1], [9/8, 5/8, 1/2, 3/2, 5/4], [45/32, 19/32, 17/32, 39/32, 5/4], [177/128, 85/128, 29/64, 93/64, 67/64], [657/512, 311/512, 271/512, 705/512, 77/64]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 3, 5}}, net cycles: 1 . order: 3

$$[0, y_1, y_3, 0, y_2]$$

$$\begin{aligned} R = & \$ [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[2/5, -3/5, 2/5], [2/5, 2/5, -3/5], [-3/5, 2/5, 2/5], [2/5, 2/5, -3/5], [2/5, -3/5, 2/5]] \$ \times \$ [[0, 2, 2, 0, 1], [0, 1, 2, 0, 2], [0, 2, 1, 0, 2]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 1 . order: 3

$$[y_1, 0, 0, y_2, y_3]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[2/5, 2/5, -3/5], [2/5, -3/5, 2/5], [2/5, 2/5, -3/5], [-3/5, 2/5, 2/5], [2/5, -3/5, 2/5]] \$ \times \$ [[2, 0, 0, 2, 1], [1, 0, 0, 2, 2], [2, 0, 0, 1, 2]] \$ \end{aligned}$$

Â» SYNC'D 13/64 , 0.2031250000

15 . Coloring, {3, 4, 5}

R: [2, 1, 4, 3, 2] **B:** [4, 3, 5, 5, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$\begin{bmatrix} -5(3+\tau) & (3+\tau^2) & 5(3+\tau) & (1+\tau) & (-3+\tau) \\ 5(9-2\tau+\tau^2) & (1+\tau) & 5(-9-5\tau-3\tau^2+\tau^3) & 5(3+\tau)^2 & (-1+\tau) \end{bmatrix}$$

For τ=1/2, [-91, -105, -99, -97, -49] . FixedPtCheck, [91, 105, 99, 97, 49]

$$\det(A + \tau \Delta) = 1(-1 + \tau)^2 (1 + \tau)^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	2 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [1/4, 0, 3/4, 0, 0], [0, 0, 0, 1/4, 3/4], [0, 0, 1/4, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-87/125, 88/75, -136/375, 1696/375, -1664/375], [273/125, -152/75, 944/375, -1184/375, \\ & 256/375], [33/125, 208/75, -776/375, 736/375, -1024/375], [-117/125, -142/75, -976/375, -64/375, \\ & 2176/375], [23/125, -2/75, 944/375, -1184/375, 256/375]] \times \$ [[1, 1/2, 1, 1, 3/2], [5/4, 5/8, 5/8, 1, \\ & 3/2], [41/32, 11/16, 23/32, 35/32, 39/32], [139/128, 5/8, 101/128, 73/64, 87/64], [301/256, 313/512, \\ & 193/256, 259/256, 741/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2}, {3, 4}}, net cycles: 2 . order: 2

$$\$ [[1, 2, 1, 1, 0], [2, 1, 1, 1, 0], [1, 2, 1, 1, 0], [2, 1, 1, 1, 0]] \$$$

$$[-y_1 + 3y_2, y_1, y_2, y_2, 0]$$

$$p = s - s^3 \quad p' = s - s^3$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, y_4]$$

$$B = \$ [[0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, -3/5, 2/5, 2/5], [1, -3/5, 2/5, -3/5], [0, 2/5, -3/5, 2/5], [0, 2/5, -3/5, 2/5], [0, 2/5, 2/5, -3/5]] \$ \times \$ [[1, 0, 1, 1, 2], [2, 0, 0, 1, 2], [2, 0, 0, 2, 1], [1, 0, 0, 2, 2]] \$$$

Â» SYNC'D 9/256 , 0.03515625000

16 . Coloring, {2, 3, 4, 5}

R: [2, 3, 4, 3, 2] **B:** [4, 1, 5, 5, 1]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$['5' ('3 + \tau')^2 ('-1 + \tau')^2, 5' ('1 + \tau') ('3 + \tau') ('-1 + \tau') ('-3 + \tau'), 5' ('1 + \tau') ('9 - \tau - \tau^2 + \tau^3'), -5' ('-9 + 2\tau - 8\tau^2 - 2\tau^3 + \tau^4'), -5' ('3 + \tau') ('-1 + \tau') ('3 + \tau^2')]$$

For τ=1/2, [49, 105, 201, 163, 91] . FixedPtCheck, [49, 105, 201, 163, 91]

$$\det(A + \tau \Delta) = 1' ('1 + \tau')^2 ('-1 + \tau')^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	3 vs 3

bi =

$$\begin{aligned} & \$ [[0, 1/4, 0, 3/4, 0], [3/4, 0, 1/4, 0, 0], [0, 0, 0, 1/4, 3/4], [0, 0, 1/4, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[7/50, 191/50, 334/75, 104/25, -928/75], [57/50, -59/50, 78/25, -296/25, 224/25], [21/25, 48/25, -32/25, 224/25, -256/25], [-24/25, -87/25, -92/25, 144/25, 64/25], [-4/25, -27/25, -196/75, -176/25, 832/75]] \$ \times \$ [[3/2, 1/2, 1/2, 1, 3/2], [3/2, 3/4, 3/8, 5/4, 9/8], [45/32, 21/32, 1/2, 39/32, 39/32], [45/32, 21/32, 15/32, 151/128, 165/128], [747/512, 345/512, 235/512, 75/64, 633/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}}, net cycles: 0 . order: 2

$$[0, y_2, y_1, y_3, 0]$$

$$R = \$ [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 1, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 0]] \$ = \$ [[1/2, -2/5, 1/10], [0, 3/5, -2/5], [0, -2/5, 3/5], [0, 3/5, -2/5], [1/2, -2/5, 1/10]] \$ \times \$ [[0, 2, 2, 1, 0], [0, 0, 3, 2, 0], [0, 0, 2, 3, 0]] \$$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 1 . order: 3

$$[y_3, 0, 0, y_1, y_2]$$

$$B = \$ [[0, 0, 0, 1, 0] , [1, 0, 0, 0, 0] , [0, 0, 0, 0, 1] , [0, 0, 0, 0, 1] , [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0] , [0, 0, 0, 0, 0] , [0, 0, 0, 0, 0] , [0, 0, 0, 1, 0] , [0, 0, 0, 0, 1]] \$ = \$ [[-3/5, 2/5, 2/5] , [2/5, 2/5, -3/5] , [2/5, -3/5, 2/5] , [2/5, -3/5, 2/5] , [2/5, 2/5, -3/5]] \$ \times \$ [[2, 0, 0, 1, 2] , [2, 0, 0, 2, 1] , [1, 0, 0, 2, 2]] \$$$

Â» SYNC'D 11/32 , 0.3437500000

SUMMARY	
Graph Type	NOT CC
v(A)	0
v(Δ)	1
π	[1, 1, 1, 1, 1]
Dbly Stoch	true

SANDWICH		Total 0
No .	Coloring	Rank

RT GROUPS		Total 1	
No .	Coloring	Rank	Solv
1	{2, 3}	5	["group", Not Solvable]

Δ-RANK'D	SC'D !RK'D	τ-RANK'D	R/B RANK'D	NOT SYNC'D	Total Runs	2 ⁿ⁻¹
15	0	15 , 15	13 , 15	1	16	16

