Lie algebras Representations and Analytic Semigroups through Dual Vector Fields

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Part IV. Polynomials

Orthogonal families

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▷ Orthogonal families

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1 Orthogonal polynomials and Fourier transform

- **Orthogonal polynomials** may be described in terms of **Fourier-Laplace** transforms.

- **Measure $p(dx)$** functions $\phi_n(x)$ are orthogonal to all polynomials of degree less than n if and only if $V_n(s)$

$$V_n(s) = \int_{-\infty}^{\infty} e^{sx}\, \phi_n(x)\, p(dx)$$

has a zero of order n at $s = 0$.

- **Follows by**

$$\left(\frac{d}{ds}\right)^k \Big|_0 V_n(s) = \int_{-\infty}^{\infty} x^k \phi_n(x)\, p(dx)$$

- **If the $\phi_n(x)$ are polynomials** they form a sequence of orthogonal polynomials.

2 Orthogonal polynomials via kernels

- **Kernels**

$$K(x, z, A)$$

forming a group under convolution

$$\int_{-\infty}^{\infty} K(x-y, z, A) K(y, z', A')\, dy = K(x, z+z', A'')$$

- **Multiplicative family**

$$\hat{K}(s, z, A) = \int_{-\infty}^{\infty} e^{sy}\, K(y, z, A)\, dy$$

Then

$$\hat{K}(s, z, A)\hat{K}(s, z,' A') = \hat{K}(s, z + z', A'')$$

- **Form the product** that integrates to $K(x, 0, A'')$, independent of z

$$K(x - y, -z, A)K(y, z, A')$$

- **Generating function for the orthogonal functions**

$$K(x-y,-z,A)K(y,z,A') = \sum z^n H_n(x,y;A,A')$$

- **By construction**

$$\int_{-\infty}^{\infty} H_n(x,y;A,A')\,dy = 0$$

- **To get orthogonality** with respect to all polynomials of degree less than n

$$\sum z^n \int_{-\infty}^{\infty} y^k H_n(x,y;A,A')\,dy$$

$$= \int_{-\infty}^{\infty} y^k K(x-y,-z,A)K(y,z,A')\,dy$$

where the terms of the summation must vanish for $k < n$.

I.e., this must reduce to a polynomial in z of degree k.

- **Or** the Fourier-Laplace transform must have terms with zeros of the corresponding order.

3 Natural exponential families

- **Means form an additive group** for a convolution family of measures.

- **The densities** provide kernels of the form $K(x, z, A)$,

where z is the mean, and A, e.g., is the variance,

or other parameters determining the distribution.

- **Gaussian distributions** $K(x, z, A) = \dfrac{e^{-(x-z)^2/(2A)}}{\sqrt{2\pi A}}$

Note that the means and variances are additive.

- **Natural exponential families** allow for parametrization by the means.

Consider MGF $M(s) = \displaystyle\int_{\mathbf{R}} e^{sx}\, p(dx)$. The NEF

$$p_s(dx) = M(s)^{-1} e^{sx}\, p(dx)$$

has means $\mu(s) = M'(s)/M(s)$.

4 Bernoulli systems

- **Bernoulli system** is a **canonical Appell system** such that the basis $\psi_n = R^n\Omega$ is orthogonal.

- **Define the generating function**

$$\omega^t(z,x) = \sum_{n\geq 0} \frac{z^n}{n!}\,\phi_n$$

where $\phi_n = n!\,\psi_n/\gamma_n$.

- **Consider a Bernoulli system in $d \geq 1$ dimensions** with canonical operator V and Hamiltonian H.

$$e^{z_\mu x_\mu - tH(z)} = \sum_{n\geq 0} \frac{V(z)^n}{n!}\,\psi_n$$

- **Fourier-Laplace transform** of ω_t times the measure of orthogonality turns out to be

$$\int e^{sy}\,\omega^t(z,y)\,p_t(dy) = e^{zV(s)+tH(s)}$$

- **Expanding in powers of z** yields the relation

$$\int_{-\infty}^{\infty} e^{sy}\,\phi_n(y)\,p_t(dy) = V(s)^n\,e^{tH(s)}$$

so that $V(0) = 0$ is all we need to conclude that the ϕ_n are an orthogonal family.

- **The function** $V(z)$ is normalized to

$V'(0) = V''(0) = 1$. And $V(0) = H(0) = 0$.

- **We take** t as our parameter A and

$$K(x, z, A) = \omega^A(z, x)p_A(x)$$

- **For** $\omega^A(z, x) \geq 0$, these are a family of probability

measures with mean $z + \mu A$, and variance $z + \sigma^2 A$,

where μ and σ^2 are the mean and variance respectively of

p_1.

- **From the basic construction**

$$K(x-y,-z,A)K(y,z,B) =$$
$$\omega^A(-z,x-y)\omega^B(z,y)\, p_A(x-y)p_B(y)$$

- $H_n(x,y;A,B) =$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\phi_k(x-y,A)\phi_{n-k}(y,B)\, p_A(x-y)p_B(y)$$

with corresponding orthogonal polynomials

$$\phi_n(x,y;A,B) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k\phi_k(x-y,A)\phi_{n-k}(y,B)$$

- **Measure of orthogonality** $p_A(x-y)p_B(y)/p_{A+B}(x)$.

- **Proof of orthogonality** is based on an addition formula for $V(s)$.

5 New families from old

For the Meixner classes, i.e., the Bernoulli systems in one variable corresponding to sl(2),

we have the corresponding classes generated as follows:

- **Gaussian** $\longrightarrow$ Gaussian
- **Poisson** $\longrightarrow$ Krawtchouk
- **Laguerre** $\longrightarrow$ Jacobi
- **Binomial** (3 types) $\longrightarrow$ Hahn (3 types)

Observe that for the binomial types, this is essentially the construction of **Clebsch-Gordan coefficients** for real forms of sl(2). This construction works for the multivariate case as well.

Probabilistically, we are looking at the distribution of X_1 given $X_1 + X_2$, where X_2 is an independent copy of X_1.

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6 Definition

- **Given a probability measure** $p(dx)$ and a

family of square-integrable functions $F(s, x)$

$$M(s) = \langle F(s, X) \rangle$$

- **Appell states** with respect to the measure p

and the family F are the functions

$$\Psi_s(x) = F(s, x)/M(s)$$

That is, the Ψ_s are the functions F normalized to have unit expectation.

- **States** comes from physics terminology denoting a function of unit norm in L^2 of p.

- **Typical choices** of the family F are

1. $F(s, x) = e^{sx}$ giving Fourier-Laplace transforms

2. $F(s, x) = (1 - sx)^{-1}$ corresponding to Stieltjes transforms.

7 Expansion in orthogonal polynomials

- **The main feature** is that the family $F(s,x)$ are eigenfunctions of an operator X_s

$$X_s\, F(s,x) = x\, F(s,x)$$

- **The family of orthogonal polynomials** is

$$\{\,\phi_n\,\} \text{ with squared norms } \gamma_n = \|\phi_n\|^2$$

- **Transforms** are defined by

$$\langle \phi_n, \Psi_s \rangle = V_n(s)$$

Thus, we have the expansion (assuming completeness)

$$\Psi_s = \sum_{n \geq 0} V_n(s) \phi_n(x)/\gamma_n$$

- **In terms of the family** F

$$F(s,x) = M(s) \sum_{n \geq 0} V_n(s) \phi_n(x)/\gamma_n$$

8 Recurrence relations for orthogonal polynomials

- **Three-term recurrence** is of the form

$$x\phi_n = c_n\phi_{n+1} + a_n\phi_n + b_n\phi_{n-1}$$

with initial conditions $\phi_{-1} = 0$, $\phi_0 = 1$.

- **The recurrence relation** implies

$$\phi_1(x) = (x - a_0)/c_0$$

- **Theorem**

Let $F(0, x) = 1$, $X_s F(s, x) = xF(s, x)$. Then

$M(s)^{-1}X_s(M(s)V_n(s)) = c_n V_{n+1} + a_n V_n + b_n V_{n-1}$

with $V_0 = 1$, $V_1 = c_0^{-1}(M^{-1}X_s M - a_0)$.

We illustrate for the Meixner case.

9 Exponential families

- **For** $F(s,x) = e^{sx}$, we have $M(s) = \langle e^{sX} \rangle$, the MGF and

$$X_s = \frac{d}{ds}$$

- **The exponential function** e^{sx} has the expansion in orthogonal polynomials

$$e^{sx} = M(s) \sum_{n \geq 0} V_n(s) \phi_n(x) / \gamma_n$$

where the coefficients $V_n, n \geq 1$,

satisfy **the recurrence formula**

$$V_n' + c_0 V_1 V_n = c_n V_{n+1} + (a_n - a_0) V_n + b_n V_{n-1}$$

with $V_0(s) = 1$ and

$$V_1(s) = c_0^{-1} \left(\frac{M'(s)}{M(s)} - a_0 \right) .$$

9.1 Meixner systems

- These arise when we have the special form

$$V_n(s) = V(s)^n$$

where, in particular, $V_1(s) = V(s)$.

- We have the expansion

$$e^{sx} = M(s) \sum_{n \geq 0} V(s)^n \phi_n(x)/\gamma_n$$

  - with $\quad V(s) = c_0^{-1}\left(\frac{M'(s)}{M(s)} - a_0\right)$

- And V satisfies the **Riccati differential equation**

$$V' = \gamma + 2\alpha V + \beta V^2$$

- The recurrence formula for the orthogonal polynomials is

$$x\phi_n = (c_0 + \beta n)\phi_{n+1} + (a_0 + 2\alpha n)\phi_n + \gamma n \phi_{n-1}$$

10 Canonical description of Meixner classes

- **Six families of orthogonal polynomials** that are canonical Appell systems.

- **The V and H operators** take the form

Meixner

$$V(z) = \frac{\tanh qz}{q - \alpha \tanh qz} \qquad H(z) = -\frac{\alpha}{\beta} z - \log \frac{qV(z)}{\sinh qz}$$

Meixner-Pollaczek	$V(z) = \tan z$	$H(z) = \log \sec z$
Krawtchouk	$V(z) = \tanh z$	$H(z) = \log \cosh z$
Charlier	$V(z) = e^z - 1$	$H(z) = e^z - 1 - z$
Laguerre	$V(z) = z/(1-z)$	$H(z) = -\log(1-z) - z$
Hermite	$V(z) = z$	$H(z) = z^2/2$

- **Parameters** are α, β with $q^2 = \alpha^2 - \beta$.

We will see how these arise by specialization from families of canonical polynomials for Lie algebras. They come from some basic Lie algebras, namely, sl(2), HW, and **osc**.

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11 Flow of the group law

- **The left dual** vector field $X^{\ddagger} = \alpha_\mu \xi_\mu^{\ddagger}$

generates the flow of the group law

$$\exp(tX^{\ddagger})f(A) = f(A(\alpha t) \odot A)$$

Setting $t = 1$ we have

$$e^{X^{\ddagger}} f(A) = f(A(\alpha) \odot A)$$

- **Let** $\hat{X} = \alpha_\mu \hat{\xi}_\mu$ be the double dual realization of X.

In terms of (x, D) variables, it is the dvf to $X^{\ddagger}$.

- **The Main Observation** for dvf's gives

$$e^{\hat{X}} e^{ax} = e^{(A(\alpha)\odot a)x}$$

- **Compare with**

$$e^{\alpha_\mu Y_\mu} e^{ax} = e^{x_\mu U_\mu (V(a)+\alpha)}$$

our main formula for dvf's in the abelian case.

11.1 Main theorem

Group elements generated by the double dual $\hat{X}$ and group elements generated by the canonical variable $\alpha_\mu Y_\mu$ give the **same result** on the vacuum state

$$e^{\hat{X}} 1 = \exp(x \cdot A(\alpha)) = e^{\alpha_\mu Y_\mu} 1 = \exp(x \cdot U(\alpha))$$

- **Correspondence** of the momentum variables with the A coordinates is

$$D \leftrightarrow A\,, \qquad V \leftrightarrow \alpha$$

Thus, the canonical operators Y_i are given as

$$Y_i = x_\mu W_{\mu i}(D)$$

where W is the inverse Jacobian matrix of the coordinate map $A \to \alpha$.

- **Express** $\partial A/d\alpha$ in the A variables, then replace every A_i by the corresponding D_i.

We have a **Lie canonical system** of polynomials $\{y_n\}$

$$e^{x \cdot A(\alpha)} = \sum_{n \geq 0} \frac{\alpha^n}{n!} y_n(x)$$

12 Cartan decomposition

- **Family of commuting self-adjoint operators** are the **quantum observables** for the system.
- We want the Lie algebra to be a **symmetric Lie algebra**

Raising operators $\mathcal{P} \longleftrightarrow$ Lowering operators $\mathcal{L}$

so that $\mathfrak{g}$ has the **Cartan decomposition**

$$\mathfrak{g} = \mathcal{L} \oplus \mathcal{K} \oplus \mathcal{P}$$

with the relations

$$[\mathcal{L}, \mathcal{P}] \subset \mathcal{K}, \qquad [\mathcal{K}, \mathcal{L}] \subset \mathcal{L}, \qquad [\mathcal{K}, \mathcal{P}] \subset \mathcal{P}$$

$\mathcal{L}$ and $\mathcal{P}$ are **abelian subalgebras** that generate $\mathfrak{g}$ as a Lie algebra.

- **Inner product** so that $L_i^* = R_i$ making $\mathfrak{g}$ a "Lie*-algebra".
- **Self-adjoint operators** of interest have the general form

$$X_i = R_i + K_i + L_i$$

A commuting family of such X_i will be the quantum observables.

13 HW

- **The coordinate map** is

$$A_1 = \alpha_1, \qquad A_2 = \alpha_2 + \tfrac{1}{2}\alpha_1\alpha_3, \qquad A_3 = \alpha_3$$

- **From the double dual**

$$\exp(\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3(R_3 + R_2 V_1))\,1 =$$

$$\exp(\alpha_1 R_1 + (\alpha_2 + \tfrac{1}{2}\alpha_1\alpha_3)R_2 + \alpha_3 R_3)\,1$$

- **Note that** R_3 and $\alpha_2 R_2$ drop out.

Setting $R_2 = t$, $\alpha_1 = \alpha_3 = z$, we get, using $R = R_1$ as our raising operator,

$$\exp(z(R + tV))\,1 = e^{zR + z^2 t/2}\,1$$

- **Our quantum observable** is $X = R + tV$, with spectral variable x. With $v = z$,

$$e^{vR}1 = e^{vx - v^2 t/2}$$

the generating function for the Hermite polynomials for the corresponding Gaussian distribution.

We have recovered our example of Chapter 1.

- **The coordinate map** is : $A_1 = \dfrac{\alpha_1 \tanh\delta}{\delta - \alpha_2 \tanh\delta}$

$$A_2 = \log\frac{\delta\,\mathrm{sech}\,\delta}{\delta - \alpha_2\tanh\delta},\ A_3 = \frac{\alpha_3\tanh\delta}{\delta - \alpha_2\tanh\delta}$$

where $\delta = \sqrt{\alpha_2^2 - \alpha_1\alpha_3}$.

- **The double dual** is : $\hat{\xi}_1 = R_1$

$$\hat{\xi}_2 = R_2 + 2R_1V_1, \quad \hat{\xi}_3 = R_3e^{2V_2} + R_2V_1 + R_1V_1^2$$

- **Now take** $\alpha_1 \to z$, $\alpha_2 \to \alpha z$, $\alpha_3 \to \beta z$, and $\delta \to qz$, $q^2 = \alpha^2 - \beta$. Noting that R_3 drops out, send $R_2 \to t$, and use $R = R_1$ as our raising operator to yield

$$e^{zX}1 = \left(\frac{q\,\mathrm{sech}\,qz}{q - \alpha\tanh qz}\right)^t \exp\left(\frac{\tanh qz}{q - \alpha\tanh qz}R\right)1$$

- **Our quantum random variable** is

$$X = R + \alpha t + 2\alpha RV + \beta(tV + RV^2)$$

- **With spectral variable x,** this is of the form

$$e^{zx} = e^{tH(z)} e^{V(z)R} 1$$

and solving for $e^{vR}1$ gives the generating function for the corresponding class of polynomials as a canonical Appell system

$$e^{vR}1 = e^{xU(v) - tH(U(v))}$$

- **Various specializations** lead to the Meixner classes for Bernoulli, negative binomial and continuous binomial (hyperbolic) distributions.

- **The gamma/exponential** family is an interesting limiting case where $q \to 0$. We get, then, with $\beta = \alpha^2$,

$$e^{zX} 1 = (1 - \alpha z)^{-t} \exp\left(R \, \frac{z}{1 - \alpha z}\right) 1$$

and solving for $z = U(v) = \dfrac{v}{1 + \alpha v}$ yields the generating function for Laguerre polynomials in an appropriate normalization.

- The Poisson and Gaussian are limiting cases as well.

15 Oscillator algebra

- For the oscillator algebra we have the coordinate map

$$A_1 = \frac{\alpha_1}{\alpha_4}(e^{\alpha_4} - 1), \quad A_2 = \alpha_2 + \frac{\alpha_1\alpha_3}{\alpha_4^2}(e^{\alpha_4} - 1 - \alpha_4),$$

$$A_3 = \frac{\alpha_3}{\alpha_4}(1 - e^{-\alpha_4}), \quad A_4 = \alpha_4.$$

- **The double dual** is

$$\hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_2, \quad \hat{\xi}_3 = R_3 + R_2V_1,$$

$$\hat{\xi}_4 = R_4 + R_1V_1 - R_3V_3$$

- **We take** $\alpha_4 \to \alpha z, \alpha_1 \to z, \alpha_3 \to \beta z,$

with R_4 dropping out and get, setting $R_2 = t$,

$$e^{zX+zY}1 =$$

$$\exp(R_1(e^{\alpha z} - 1)/\alpha) \quad \cdot$$

$$\exp(\beta t(e^{\alpha z} - 1 - \alpha z)/\alpha^2)\exp(R_3\beta(1 - e^{-\alpha z})/\alpha)\,1$$

- **where**

$$X = R_1 + \alpha R_1 V_1 + \beta t V_1$$

and

$$Y = \beta R_3 - \alpha R_3 V_3$$

- **The Y term** gives an independent aff(2).

- **The X term** gives, with $R \to R_1$,

$$e^{vR} 1 = (1 + \alpha v)^{x/\alpha + \beta t/\alpha^2} \exp(-v\beta t/\alpha)$$

which is the generating function for **Poisson-Charlier polynomials** for a scaled Poisson process with drift.

- **Observe** that in each case, the formula for X in terms of R and V gives the **three-term recurrence** relation for the corresponding orthogonal polynomials.

16 Procedure

1. Cartan decomposition.

2. Find coordinate mapping.

3. Exponentiate the double dual.

4. Arrange into commuting operators of the form
$X = R + K + L$.

5. Group elements generated by raising operators acting on the vacuum are generating functions for the basis of the representation.

6. Expectation values of group elements generated by X operators interpreted as moment generating functions yield the spectral measures.