Lie algebras Representations and Analytic Semigroups through Dual Vector Fields

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Part III. Dual Vector Fields

DVFs

Flow of a dual vector field

Canonical polynomials

1 Vector fields and their duals

♢ **Recall the double dual** in (x, D) variables.
They are operators of the form

$$x_\mu W_\mu(D)$$

♢ **Main Observation** are the relations

$$x \cdot W(D)\, e^{A\cdot x} = W(A) \cdot \partial_A\, e^{A\cdot x} = x \cdot W(A)\, e^{A\cdot x}$$

with $\partial_A = (\partial_1, \partial_2, \ldots, \partial_d)$, $\partial_i = \partial/\partial A_i$.

What this does is exchange the operators in the (x, D) variables with corresponding operators in (A, ∂_A) variables, effectively as an **algebraic Fourier transform**, interchanging derivatives and variables.

Thus each vector field has its dual and vice versa.

♢ **dual vector field** is "dvf" for short. Write

$$Y_i = x_\mu W_{\mu i}(D), \quad \tilde{Y}_i = W_{\mu i}(A)\partial_\mu$$

We will be interested in families of
commuting vector fields.
We will use a "canonical" construction.

2 Canonical coordinates

For an operator $f(D)$, the function $f(z)$ is referred to as its **symbol**.

◇ **Canonical coordinates** Start with a function $V(z) = (V_1(z), \ldots, V_N(z))$ holomorphic in a neighborhood of 0, with $V(0) = 0$, and the Jacobian matrix $V' = \left(\frac{\partial V_i}{\partial z_j}\right)$ nonsingular at 0.

$U(v)$ denotes the functional inverse of V, i.e., $z_j = U_j(V(z))$.

The components V_i are **canonical coordinates** or canonical functions.

◇ **Canonical variables** are associated dvf's

$$Y_j = x_\lambda W_{\lambda j}(D)$$

where $W(z) = V'(z)^{-1}$ is the matrix inverse to $V'(z)$.

3 Raising operators

◇ **Commutation relations** $[V_i(D), Y_j] = \delta_{ij} I$.

◇ Proof: We have

$$[V_i(D), x_\lambda] W_{\lambda j} = (V')_{i\lambda} W_{\lambda j} = \delta_{ij} I$$

◇ **That the Y's commute** follows ultimately from equality of the mixed partials of V.

From now on,
W will refer to the inverse Jacobian of a given function V.

♩ If V is a linear mapping, $V_i(z) = S_{i\lambda} z_\lambda$, where S is an invertible constant matrix, with inverse T, we have $V' = S$. Thus

$$V_i(D) = S_{i\lambda} D_\lambda, \qquad Y_i = x_\mu T_{\mu i}$$

The inverse function is $U_i(v) = T_{i\lambda} v_\lambda$.

♪ For the Poisson case, with $V(z) = e^z - 1$, we have $W(z) = e^{-z}$ and $U(v) = \log(1 + v)$.

4 Flow of a DVF

◇ **Given the dvf** $\quad Y = v_\lambda Y_\lambda = v_\lambda x_\mu W_{\mu\lambda}(D),$

we wish to calculate $\dfrac{\partial u}{\partial t} = Yu, \quad u(0) = f(x).$

From the Main Observation we have

$$e^{tY} e^{A\cdot x} = e^{t\tilde{Y}} e^{A\cdot x} = e^{A(t)\cdot x}$$

For the vector field $\tilde{Y}$ we have: $\quad \dot{A}_i = v_\lambda W_{i\lambda}(A)$

Multiplying both sides by $V'(A)$ yields $\dot{A}_\mu V'_{k\mu} = v_k$.

Now the left-hand side is an exact derivative. I.e.,

$$\frac{d}{dt} V_k(A(t)) = v_k$$

Integrating, with initial conditions $A(0) = A$, we get

$$V(A(t)) = V(A) + tv$$

Solving, we have

$$A(t) = U(tv + V(A))$$

5 Main Formula

◇ Main Formula

$$e^{tY}e^{A\cdot x}=e^{t\tilde{Y}}e^{A\cdot x}=e^{x\cdot U(tv+V(A))}$$

A useful corollary is the action of the dvf Y on the vacuum function equal to 1. We get this by setting $A=0$:

$$e^{tY}1=e^{x\cdot U(tv)}$$

For one variable we have

◇ Main Formula for $d=1$

For a canonical function $V(z)$,
with $W(z)=1/V'(z), U(V(z))=z$.
Let $Y=xW(D)$ be the associated canonical variable.
Then we have

$$e^{vY}e^{Ax}=e^{xU(v+V(A))}$$

And, in particular,

$$e^{vY}1=e^{xU(v)}$$

DVFs

Flow of a dual vector field

▷ Canonical polynomials

6 Recurrence formula

◇ **Canonical polynomials** are the basis for our vector space.

$$y_n(x) = Y^n 1$$

Note that the vacuum is the constant function equal to 1. The raising operator is Y, lowering operator $V = V(D)$

$$Y y_n(x) = y_{n+1}(x), V(D) y_n(x) = n y_{n-1}(x)$$

providing a representation of the HW algebra.

The generating function for the canonical polynomials is

$$e^{v \cdot Y} 1 = e^{x \cdot U(v)} = \sum_{n \geq 0} \frac{v^n}{n!} y_n(x)$$

♪ For $V(z) = e^z - 1$,

$$Y = xe^{-D}$$

The shift operator $e^{-D} f(x) = f(x-1)$ so

$$\begin{aligned} y_n(x) &= Y^n 1 = xe^{-D} y_{n-1}(x) \\ &= x(x-1)\cdots(x-n+1) = x^{(n)} \end{aligned}$$

the n^{th} factorial power.

With $U(v) = \log(1+v)$, the expansion is

$$(1+v)^x = \sum_{n\geq 0} \frac{v^n}{n!} x^{(n)}$$

the standard binomial theorem.

7 Random walk formula

- **Moment generating function** $W(z) = \sum_{n \geq 0} \frac{z^n}{n!} \mu_n$
  - Define generalized moments

$$\langle\langle X^n \rangle\rangle = \mu_n$$

  - Probabilistic case

$$\langle X^n \rangle = \mu_n$$

- For an analytic function f, expand

$$f(x + X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} f^{(n)}(x)$$

where here X denotes a virtual or actual random variable. $W(D)$ acts as a **formal convolution operator**

$$W(D)\, f(x) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} f^{(n)}(x) = \langle\langle f(x + X) \rangle\rangle$$

We extend the generalized averaging to several variables by taking them to be effectively independent:

$$\langle\langle X_1^{n_1} X_2^{n_2} \ldots X_m^{n_m} \rangle\rangle = \mu_{n_1} \mu_{n_2} \cdots \mu_{n_m}$$

$\diamondsuit$ **Random walk formula** The canonical polynomials may be expressed in the form of 'generalized factorials.'

$$y_n(x)$$

$$= \langle\!\langle x(x+X_1)(x+X_1+X_2)\cdots(x+X_1+X_2+\cdots+X_{n-1})\rangle\!\rangle$$

$\diamondsuit$ **Random walk** $S_n = X_1 + X_2 + \cdots + X_n$, where the X_i are independent, identically distributed random variables with moment generating function equal to W.

- $\diamondsuit$ With $S_0 = x$, the corresponding expectation value is denoted by $\langle\cdot\rangle_x$.

Then

$$y_n(x) = \langle S_0 S_1 S_2 \cdots S_{n-1}\rangle_x$$

Note that this is the product of consecutive variables of the random walk.

◇ **In the probabilistic case** write

$$W(D) = \int e^{uD}\, p(du)$$

Then

$$(xW(D))^n = x \int e^{u_1 D}\, p(du_1) \cdots x \int e^{u_n D}\, p(du_n)$$

With $e^{uD} f(x) = f(x+u) e^{uD}$, we get

$$\begin{aligned}
&(xW(D))^n \\
&= \int x(x+u_1)(x+u_1+u_2)\cdots(x+u_1+\cdots+u_{n-1}) \\
&\cdot \exp\big((\sum_{j=1}^{n} u_j)D\big)\, p(du_1)\cdots p(du_n)
\end{aligned}$$

◇ This is a **formula for the operator** Y^n

$$Y^n = \langle S_0 S_1 S_2 \cdots S_{n-1} e^{S_n D} \rangle_x$$

Thus the expansion

$$e^{xU(v)} = 1 + x \sum_{n=0}^{\infty} \frac{v^n}{n!} \langle \prod_{j=1}^{n-1} (x + S_j) \rangle_0$$

♩ Exponential random walk and Bessel polynomials

Exponential distribution with mean q has $W(z) = (1 - qz)^{-1}$ or

$$V = z - qz^2/2\,, \qquad U = \frac{1 - \sqrt{1 - 2qv}}{q}$$

♢ Let $T_1, T_2, \ldots, T_n, \ldots$ be independent exponentials with mean q. Then

$$\langle T_1(T_1{+}T_2)\cdots(T_1{+}T_2{+}\cdots{+}T_n)\rangle = n!\binom{2n}{n}\left(\frac{q}{2}\right)^n$$

♢ **Consider** $\quad V = z - z^2/2, U = 1 - \sqrt{1 - 2v}$.

From the classical theory of random walks we have

$$\frac{(1 - \sqrt{1 - 2v})^n}{\sqrt{1 - 2v}} = \sum_{p \geq 0} \frac{v^{n+p}}{2^p}\binom{n + 2p}{p}$$

Multiplying by $x^n/n!$ and summing gives the generating function for Bessel polynomials $\theta_n(x)$:

$$\frac{1}{\sqrt{1 - 2v}} e^{x(1 - \sqrt{1 - 2v})}$$

Differentiating $e^{x(1-\sqrt{1-2v})}$ with respect to v and integrating back we find

$$e^{x(1-\sqrt{1-2v})} = 1+\sum_{n\geq 1} \frac{x^n}{n!} \sum_{p\geq 0} \frac{n}{n+p} \frac{v^{n+p}}{2^p} \binom{n+2p-1}{p}$$

Thus, we have

$$y_n(x) = \sum_p \binom{n+2p-1}{p} 2^p (\tfrac{1}{2})_p \, x^{n-p}$$

♪ Cayley example

With $V(z) = z\,e^{-z}$, we get $W(z) = e^z(1-z)^{-1}$, so that the corresponding probability distribution is an exponential with mean 1 shifted by 1.

Checking that

$$y_n(x) = x(x+n)^{n-1}$$

we find

$$n^{n-1} = \langle (1+T_1)(2+T_1+T_2)\cdots(n-1+T_1+T_2+\cdots+T_{n-1}) \rangle$$

9 Inversion of analytic functions

◇ **Expanding in powers of x** we have

$$e^{xU(v)} = \sum_{m \geq 0} \frac{x^m}{m!} (U(v))^m$$

Thus

the coefficient of $x^m/m!$ in $y_n(x)$ gives
the coefficient of $v^n/n!$ in the expansion of $U(v)^m$

◇ **Applying the operator $g(D)$** and evaluating at $x = 0$

$$g(U(v)) = \sum_{n \geq 0} \frac{v^n}{n!} g(D) y_n(0)$$

◇ **Expansion of $U(v)$** is the coefficient of x

$$U(v) = \sum_{n \geq 0} \frac{v^n}{n!} y_n'(0)$$

◇ **In the random walk formulation**

$$U(v) = \sum_{n=1}^{\infty} \frac{v^n}{n!} \langle \prod_{j=1}^{n-1} S_j \rangle_0$$

10 Examples

♪ Inverse distribution

Given an analytic moment generating function $W(z)$, we can form

$$V(z) = \int_0^z \frac{d\zeta}{W(\zeta)}$$

And the inverse of V is given by the above formula. In particular, if $V'(x)$ is a density function, we have the expansion for the inverse distribution function.

♫ Inverse Gaussian distribution and Gaussian random walk

With $W(z) = e^{z^2/2}$, we get V as the distribution function of a standard Gaussian, modulo a factor of $\sqrt{2\pi}$. Thus, we have the expansion of the inverse Gaussian distribution in terms of

(i) the values $y_n'(0)$

or

(ii) in terms of the Gaussian random walk.

11 Dual approach using vector fields

◇ **From our Main Observation** we have

$$e^{vY}e^{Ax} = e^{v\tilde{Y}}e^{Ax} \quad = \quad e^{xU(v+V(A))}$$

$$= \quad \sum_{m\geq 0} \frac{(v+V(A))^m}{m!} y_m(x)$$

◇ **Iterating**

$$\text{application of } Y \quad \leftrightarrow \quad \frac{d}{dv} \quad \leftrightarrow \quad \text{application of } \tilde{Y}$$

n times we get

$$(\tilde{Y})^n e^{Ax} = \sum_{m\geq 0} \frac{V(A)^m}{m!} y_{m+n}(x)$$

the action of $\tilde{Y}^n$ on the exponential.

◇ **Recover $y_n(x)$** by setting $A = 0$

$$y_n(x) = (W(A)\partial_A)^n e^{Ax}\Big|_{A=0}$$

♢ **The flow of the vector field** $\tilde{Y}$ on $g(A)$ is

$$e^{v\tilde{Y}} g(A) = g\big(U(v + V(A))\big)$$

♢ **Letting** $A = 0$

$$e^{v\tilde{Y}} g(0) = g(U(v)))$$

Or

$$g(U(v)) = \sum_{n \geq 0} \frac{v^n}{n!} \tilde{Y}^n g(0)$$

(approach suggested by D. Dominici)

♢ **For** $g(D) = D$ we have

$$(\tilde{Y})^n A \Big|_{A=0} = (\tilde{Y})^{n-1} W(0)$$

which gives the coefficient of $v^n/n!$ in the expansion of $U(v)$.

12 Example

♫ For $V(z) = 1 - e^{-z}$, $\quad \tilde{Y} = e^A \partial_A$.
So $U(v) = -\log(1-v)$ and

$$U(v)^m = \sum_{n \geq 0} \frac{v^n}{n!} (e^A \partial_A)^n A^m \Big|_{A=0}$$

On the other hand,

$y_n(x) = Y^n 1 =$

$(xe^D)^n 1 = x(x+1) \cdots (x+n-1) = (x)_n = \sum_k S_{nk} x^k$

S_{nk} are absolute values of Stirling numbers of the first kind. And

$$D^m y_n(0) = m!\, S_{nm}$$

So

$$\begin{aligned} (-\log(1-v))^m &= \sum_{n \geq 0} \frac{v^n}{n!} (e^A \partial_A)^n A^m \Big|_{A=0} \\ &= \sum_{n \geq 0} \frac{v^n}{n!} m!\, S_{nm} \end{aligned}$$

another variation on the binomial theorem as seen by expanding $(1-v)^{-x}$.

13 Concluding remarks

Our approach in this part applies equally well in d variables,

with n as multi-index and $Y^n = Y_1^{n_1} \cdots Y_d^{n_d}$,

as it is based on the Main Observation which holds in all dimensions.

The essential feature is that $Y = v_\lambda Y_\lambda$, where Y_i generate an abelian algebra.