SU(2)

```
> X:=matrix(2,2,[a[2],a[1],-a[3],-a[2]]);
```

$$X := \begin{bmatrix} a_2 & a_1 \\ -a_3 & -a_2 \end{bmatrix}$$

```
> d:=3:n:=2:lie(X,d,n,0);kirlv();duals();
```

```
`the group element is computed`
`solving equations for pi-dagger`
`solving equations for pi-star`
`piadj available`
`group element`
```

$$\begin{bmatrix} e^{A_2} - A_1 A_3 e^{-A_2} & A_1 e^{-A_2} \\ -A_3 e^{-A_2} & e^{-A_2} \end{bmatrix}$$

```
`left-dual`
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 2\,A_1 & 1 & 0 \\ {A_1}^2 & A_1 & e^{2\,A_2} \end{bmatrix}$$

```
`right-dual`
```

$$\begin{bmatrix} e^{2\,A_2} & A_3 & {A_3}^2 \\ 0 & 1 & 2\,A_3 \\ 0 & 0 & 1 \end{bmatrix}$$

$KIRLV$

$$\begin{bmatrix} 0 & -2\,x_1 & -x_2 \\ 2\,x_1 & 0 & -2\,x_3 \\ x_2 & 2\,x_3 & 0 \end{bmatrix}$$

DUALS

`xi-dagger`$_1 = \delta_1$
`xi-dagger`$_2 = 2\,A_1\delta_1 + \delta_2$
`xi-dagger`$_3 = {A_1}^2\delta_1 + A_1\delta_2 + e^{2\,A_2}\delta_3$
` `
`-------------------------------------------------------- `
` `
`xi-star`$_1 = e^{2\,A_2}\delta_1 + A_3\delta_2 + {A_3}^2\delta_3$
`xi-star`$_2 = \delta_2 + 2\,A_3\delta_3$
`xi-star`$_3 = \delta_3$
` `
`--------------------------------------------------------- `
` `
`xi-hat`$_1 = R_1$
`xi-hat`$_2 = 2\,V_1R_1 + R_2$
`xi-hat`$_3 = {V_1}^2R_1 + V_1R_2 + e^{2\,V_2}R_3$

Heisenberg-Weyl HW

```
> X:=matrix(3,3,[0,a[3],a[2],0,0,a[1],0,0,0]):
> d:=3;n:=3;lie(X,d,n,0):kirlv();duals();
```

$d := 3$

$n := 3$

`the group element is computed`
`solving equations for pi-dagger`
`solving equations for pi-star`
`piadj available`
`group element`

$$\begin{bmatrix} 1 & A_3 & A_2 \\ 0 & 1 & A_1 \\ 0 & 0 & 1 \end{bmatrix}$$

`left-dual`

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_1 & 1 \end{bmatrix}$$

`right-dual`

$$\begin{bmatrix} 1 & A_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$KIRLV$

$$\begin{bmatrix} 0 & 0 & -x_2 \\ 0 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}$$

$DUALS$

‘xi-dagger‘$_1 = \delta_1$
‘xi-dagger‘$_2 = \delta_2$
‘xi-dagger‘$_3 = A_1\delta_2 + \delta_3$
‘ ‘
‘-- ‘
‘ ‘
‘xi-star‘$_1 = \delta_1 + A_3\delta_2$
‘xi-star‘$_2 = \delta_2$
‘xi-star‘$_3 = \delta_3$
‘ ‘
‘-- ‘
‘ ‘
‘xi-hat‘$_1 = R_1$
‘xi-hat‘$_2 = R_2$
‘xi-hat‘$_3 = V_1R_2 + R_3$

Affine aff(2)

```
> X:=matrix(2,2,[[a[2],a[1]],[0,0]]);
```

$$X := \begin{bmatrix} a_2 & a_1 \\ 0 & 0 \end{bmatrix}$$

```
> d:=2;n:=2;lie(X,d,n,0);kirlv();duals();
```

$d := 2$
$n := 2$

```
`the group element is computed`
`solving equations for pi-dagger`
`solving equations for pi-star`
`piadj available`
`group element`
```

$$\begin{bmatrix} e^{A_2} & A_1 \\ 0 & 1 \end{bmatrix}$$

```
`left-dual`
```

$$\begin{bmatrix} 1 & 0 \\ A_1 & 1 \end{bmatrix}$$

```
`right-dual`
```

$$\begin{bmatrix} e^{A_2} & 0 \\ 0 & 1 \end{bmatrix}$$

$KIRLV$

$$\begin{bmatrix} 0 & -x_1 \\ x_1 & 0 \end{bmatrix}$$

DUALS

`xi-dagger`$_1 = \delta_1$
`xi-dagger`$_2 = A_1\delta_1 + \delta_2$
‘ ‘
‘-- ‘
‘ ‘
`xi-star`$_1 = e^{A_2}\delta_1$
`xi-star`$_2 = \delta_2$
‘ ‘
‘-- ‘
‘ ‘
`xi-hat`$_1 = R_1$
`xi-hat`$_2 = V_1R_1 + R_2$

Euclidean: E2

```
> X:=matrix(3,3,[[0,a[3],a[1]],[-a[3],0,(a[2])],[0,0,0]]);
```

$$X := \begin{bmatrix} 0 & a_3 & a_1 \\ -a_3 & 0 & a_2 \\ 0 & 0 & 0 \end{bmatrix}$$

```
> d:=3:n:=3:lie(X,d,n,1);kirlv();duals();
```

```
'the group element is computed'
'solving equations for pi-dagger'
'solving equations for pi-star'
'piadj available'
'group element'
```

$$\begin{bmatrix} \cos(A_3) & \sin(A_3) & A_1 \\ -\sin(A_3) & \cos(A_3) & A_2 \\ 0 & 0 & 1 \end{bmatrix}$$

```
'left-dual'
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A_2 & -A_1 & 1 \end{bmatrix}$$

```
'right-dual'
```

$$\begin{bmatrix} \cos(A_3) & -\sin(A_3) & 0 \\ \sin(A_3) & \cos(A_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$KIRLV$

$$\begin{bmatrix} 0 & 0 & x_2 \\ 0 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

DUALS

‘`xi-dagger`‘$_1 = \delta_1$
‘`xi-dagger`‘$_2 = \delta_2$
‘`xi-dagger`‘$_3 = A_2\delta_1 - A_1\delta_2 + \delta_3$
‘ ‘
‘`-----------------------------------------------------` ‘
‘ ‘
‘`xi-star`‘$_1 = \cos(A_3)\,\delta_1 - \sin(A_3)\,\delta_2$
‘`xi-star`‘$_2 = \sin(A_3)\,\delta_1 + \cos(A_3)\,\delta_2$
‘`xi-star`‘$_3 = \delta_3$
‘ ‘
‘`------------------------------------------------------` ‘
‘ ‘
‘`xi-hat`‘$_1 = R_1$
‘`xi-hat`‘$_2 = R_2$
‘`xi-hat`‘$_3 = V_2R_1 - V_1R_2 + R_3$

Oscillator osc

```
> X:=matrix(3,3,[0,a[3],a[2],0,a[4],a[1],0,0,0]);d:=4;n:=3;lie(X,d,n,0):
```

```
> kirlv();duals();
```

$$X := \begin{bmatrix} 0 & a_3 & a_2 \\ 0 & a_4 & a_1 \\ 0 & 0 & 0 \end{bmatrix}$$

$d := 4$
$n := 3$

`the group element is computed`
`solving equations for pi-dagger`
`solving equations for pi-star`
`piadj available`
`group element`

$$\begin{bmatrix} 1 & A_3 e^{A_4} & A_2 \\ 0 & e^{A_4} & A_1 \\ 0 & 0 & 1 \end{bmatrix}$$

`left-dual`

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & A_1 & 1 & 0 \\ A_1 & 0 & -A_3 & 1 \end{bmatrix}$$

`right-dual`

$$\begin{bmatrix} e^{A_4} & A_3 e^{A_4} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{-A_4} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$KIRLV$

$$\begin{bmatrix} 0 & 0 & -x_2 & -x_1 \\ 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & x_3 \\ x_1 & 0 & -x_3 & 0 \end{bmatrix}$$

$DUALS$

‘xi-dagger‘$_1 = \delta_1$
‘xi-dagger‘$_2 = \delta_2$
‘xi-dagger‘$_3 = A_1\delta_2 + \delta_3$
‘xi-dagger‘$_4 = A_1\delta_1 - A_3\delta_3 + \delta_4$
‘ ‘
‘-- ‘
‘ ‘
‘xi-star‘$_1 = e^{A_4}\delta_1 + A_3e^{A_4}\delta_2$
‘xi-star‘$_2 = \delta_2$
‘xi-star‘$_3 = e^{-A_4}\delta_3$
‘xi-star‘$_4 = \delta_4$
‘ ‘
‘-- ‘
‘ ‘
‘xi-hat‘$_1 = R_1$
‘xi-hat‘$_2 = R_2$
‘xi-hat‘$_3 = V_1R_2 + R_3$
‘xi-hat‘$_4 = V_1R_1 - V_3R_3 + R_4$

Schrödinger

```
> d:=6;n:=4;
> X := matrix([[0, a[2], a[3], -2*a[1]], [0, a[5], a[6], a[3]],
[0, -a[4], -a[5], -a[2]], [0, 0, 0, 0]]);
```

$$X := \begin{bmatrix} 0 & a_2 & a_3 & -2\,a_1 \\ 0 & a_5 & a_6 & a_3 \\ 0 & -a_4 & -a_5 & -a_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> lie(X,d,n,0);kirlv();duals();
‘the group element is computed‘
‘solving equations for pi-dagger‘
‘solving equations for pi-star‘
‘piadj available‘
```

`group element`

$$\begin{bmatrix} 1 & e^{A_5}A_2 - e^{A_5}A_3A_4 & e^{A_5}A_6A_2 - e^{A_5}A_6A_3A_4 + \frac{A_3}{e^{A_5}} & A_2A_3 - 2\,A_1 \\ 0 & e^{A_5} & e^{A_5}A_6 & A_3 \\ 0 & -A_4e^{A_5} & -A_4e^{A_5}A_6 + \left(e^{A_5}\right)^{-1} & -A_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

`left-dual`

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ A_2 & 0 & 1 & 0 & 0 & 0 \\ 1/2\,{A_3}^2 & A_3 & 0 & 1 & 0 & 0 \\ 0 & -A_2 & A_3 & -2\,A_4 & 1 & 0 \\ -1/2\,{A_2}^2 & 0 & -A_2 & {A_4}^2 & -A_4 & e^{-2\,A_5} \end{bmatrix}$$

`right-dual`

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -A_3\left(A_4e^{2\,A_5}A_6 - 1\right)e^{-A_5} & -\left(A_4e^{2\,A_5}A_6 - 1\right)e^{-A_5} & -e^{A_5}A_6 & 0 & 0 & 0 \\ e^{A_5}A_3A_4 & A_4e^{A_5} & e^{A_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2\,A_5} & -A_6 & {A_6}^2 \\ 0 & 0 & 0 & 0 & 1 & -2\,A_6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$KIRLV$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_1 & 0 & x_2 & x_3 \\ 0 & x_1 & 0 & -x_2 & -x_3 & 0 \\ 0 & 0 & x_2 & 0 & 2\,x_4 & x_5 \\ 0 & -x_2 & x_3 & -2\,x_4 & 0 & 2\,x_6 \\ 0 & -x_3 & 0 & -x_5 & -2\,x_6 & 0 \end{bmatrix}$$

DUALS

$`xi-dagger`_1 = \delta_1$
$`xi-dagger`_2 = \delta_2$
$`xi-dagger`_3 = A_2\delta_1 + \delta_3$
$`xi-dagger`_4 = 1/2\, {A_3}^2\delta_1 + A_3\delta_2 + \delta_4$
$`xi-dagger`_5 = -A_2\delta_2 + A_3\delta_3 - 2\, A_4\delta_4 + \delta_5$
$`xi-dagger`_6 = -1/2\, {A_2}^2\delta_1 - A_2\delta_3 + {A_4}^2\delta_4 - A_4\delta_5 + e^{-2\,A_5}\delta_6$

` `
`------------------------------------------------------ `
` `

$`xi-star`_1 = \delta_1$
$`xi-star`_2 = -A_3\left(A_4e^{2\,A_5}A_6 - 1\right)e^{-A_5}\delta_1 - \left(A_4e^{2\,A_5}A_6 - 1\right)e^{-A_5}\delta_2 - e^{A_5}A_6\delta_3$
$`xi-star`_3 = e^{A_5}A_3A_4\delta_1 + A_4e^{A_5}\delta_2 + e^{A_5}\delta_3$
$`xi-star`_4 = e^{-2\,A_5}\delta_4 - A_6\delta_5 + {A_6}^2\delta_6$
$`xi-star`_5 = \delta_5 - 2\,A_6\delta_6$
$`xi-star`_6 = \delta_6$

` `
`------------------------------------------------------ `
` `

$`xi-hat`_1 = R_1$
$`xi-hat`_2 = R_2$
$`xi-hat`_3 = V_2R_1 + R_3$
$`xi-hat`_4 = 1/2\, {V_3}^2R_1 + V_3R_2 + R_4$
$`xi-hat`_5 = -V_2R_2 + V_3R_3 - 2\,V_4R_4 + R_5$
$`xi-hat`_6 = -1/2\, {V_2}^2R_1 - V_2R_3 + {V_4}^2R_4 - V_4R_5 + e^{-2\,V_5}R_6$

Finite Difference FD

```
> X:=matrix(2,2,[[a[3],a[2]],[0,a[1]]]);
```

$$X := \begin{bmatrix} a_3 & a_2 \\ 0 & a_1 \end{bmatrix}$$

```
> d:=3:n:=2:lie(X,d,n,0);kirlv();duals();
```

‘the group element is computed‘
‘solving equations for pi-dagger‘
‘solving equations for pi-star‘
‘piadj available‘
‘group element‘

$$\begin{bmatrix} e^{A_3} & A_2 \\ 0 & e^{A_1} \end{bmatrix}$$

‘left-dual‘

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{A_1} & 0 \\ 0 & A_2 & 1 \end{bmatrix}$$

‘right-dual‘

$$\begin{bmatrix} 1 & A_2 & 0 \\ 0 & e^{A_3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$KIRLV$

$$\begin{bmatrix} 0 & -x_2 & 0 \\ x_2 & 0 & -x_2 \\ 0 & x_2 & 0 \end{bmatrix}$$

$DUALS$

‘xi-dagger‘$_1 = \delta_1$
‘xi-dagger‘$_2 = e^{A_1}\delta_2$
‘xi-dagger‘$_3 = A_2\delta_2 + \delta_3$
‘ ‘
‘--- ‘
‘ ‘
‘xi-star‘$_1 = \delta_1 + A_2\delta_2$
‘xi-star‘$_2 = e^{A_3}\delta_2$
‘xi-star‘$_3 = \delta_3$
‘ ‘
‘-- ‘
‘ ‘
‘xi-hat‘$_1 = R_1$
‘xi-hat‘$_2 = e^{V_1}R_2$
‘xi-hat‘$_3 = V_2R_2 + R_3$

```
> LagrangeW(exp(z^2/2),10);
```

$$x$$

$$x^2$$

$$x^3 + x$$

$$x^4 + 4\,x^2$$

$$x^5 + 10\,x^3 + 7\,x$$

$$x^6 + 20\,x^4 + 52\,x^2$$

$$x^7 + 35\,x^5 + 217\,x^3 + 127\,x$$

$$x^8 + 56\,x^6 + 672\,x^4 + 1408\,x^2$$

$$x^9 + 84\,x^7 + 1722\,x^5 + 8380\,x^3 + 4369\,x$$

$$x^{10} + 120\,x^8 + 3864\,x^6 + 35680\,x^4 + 65104\,x^2$$

"U="

$$v + \frac{1}{6}\,v^3 + \frac{7}{120}\,v^5 + \frac{127}{5040}\,v^7 + \frac{4369}{362880}\,v^9$$

```
> v:=.3024;evalf(U);
```

$$v := .3024$$

$$.3071624593$$

```
> int(exp(-u^2/2),u=0..%);
```

.3023999881