Lie algebras Representations and Analytic Semigroups through Dual Vector Fields

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Part V. Jacobians

1 Main formula

⋆ **Denote the transpose** of $\pi^\ddagger$ by $\hat{\pi}$, of π^* by $\hat{\pi}^*$.

⋆ **The exponential** of the adjoint representation, Ad_g , connects the left and right duals

$$g\xi_j = g\xi_j g^{-1} g = \xi_j^* g = Ad_g(\xi_j) g = Ad_g(\xi_j^\ddagger) g$$

⋆ Ad_g acts on the row vector with components $\xi_i^\ddagger$.

⋆ **Solving** for ∂_i shows that the left and right duals are related as columns by

$$\xi^* = \pi^* {\pi^\ddagger}^{-1} \xi^\ddagger$$

⋆ **Define the adjoint group** element

$$\breve{\pi} = g(A, \breve{\xi})$$

⋆ The **exponential of the adjoint representation** is given by

$$\breve{\pi} = \hat{\pi}^{-1} \hat{\pi}^*$$

Proof: As columns

$$\xi^* = \breve{\pi}^\dagger \xi^\ddagger$$

And we see that $\breve{\pi}^\dagger = \pi^* {\pi^\ddagger}^{-1}$ and hence the result.

2 Example

♪ For the affine group we have

$$\check{\xi}_1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \qquad \check{\xi}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Calculating exponentials gives

$$e^{A_1\check{\xi}_1} e^{A_2\check{\xi}_2} = \begin{pmatrix} 1 & -A_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{A_2} & -A_1 \\ 0 & 1 \end{pmatrix}$$

Recalling the pi-matrices

$$\pi^\ddagger = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}, \qquad \pi^* = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

it is readily checked that this is the transpose of $\pi^* {\pi^\ddagger}^{-1}$.

Adjoint group

▷ Jacobians of the group law

Three classes of operators for any Lie algebra

Conclusion

3 Group law and pi-matrices

⋆ **Write** $g = g(A)g(B) = g(C)$, with $C = A \odot B$.

⋆ **Differentiating** with respect to A_i we get

$$\Pi^{\ddagger}_{i\mu}(A)\,\xi^{\ddagger}_{\mu}(A)g = \frac{\partial g}{\partial C_\lambda}\frac{\partial C_\lambda}{\partial A_i} = \Pi^{\ddagger}_{i\mu}(A)\,\xi_\mu g$$

⋆ And

$$\Pi^{*}_{i\mu}(B)\,\xi^{*}_{\mu}(B)g = \frac{\partial g}{\partial C_\lambda}\frac{\partial C_\lambda}{\partial B_i} = \Pi^{*}_{i\mu}(B)\,g\xi_\mu$$

⋆ So

$$\begin{aligned}\frac{\partial g}{\partial C_i} &= \Pi^{\ddagger}_{i\mu}(C)\,\xi^{\ddagger}_{\mu}(C)g = \Pi^{\ddagger}_{i\mu}(C)\,\xi_\mu g \\ &= \Pi^{*}_{i\mu}(C)\,\xi^{*}_{\mu}(C)g = \Pi^{*}_{i\mu}(C)\,g\xi_\mu\end{aligned}$$

⋆ **Solving** writing $C = A \odot B$ yields

$$\frac{\partial (A \odot B)}{\partial A} = \hat{\pi}(A \odot B)\, \hat{\pi}^{-1}(A)$$

$$\frac{\partial (A \odot B)}{\partial B} = \hat{\pi}^{*}(A \odot B)\, \hat{\pi}^{*-1}(B)$$

⋆ **Letting** $A = 0$ in the first equation, $B = 0$ in the second yields

$$\left.\frac{\partial (A \odot B)}{\partial A}\right|_{A=0} = \hat{\pi}(B)$$

$$\left.\frac{\partial (A \odot B)}{\partial B}\right|_{B=0} = \hat{\pi}^{*}(A)$$

4 Examples

$\boxed{♫}$ Recall the HW group law : $C_1 = A_1 + B_1$

$$C_2 = A_2 + B_2 + A_3 B_1, \quad C_3 = A_3 + B_3$$

and the pi-matrices

$$\pi^{\ddagger}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_1 & 1 \end{pmatrix}, \qquad \pi^{*}(A) = \begin{pmatrix} 1 & A_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Calculating the Jacobians, we find

$$\frac{\partial C_i}{\partial A_j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & B_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \frac{\partial C_i}{\partial B_j} = \begin{pmatrix} 1 & 0 & 0 \\ A_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that in this case the pi-matrices are a representation of the group, i.e.

$$\hat{\pi}(A \odot B) = \hat{\pi}(A)\hat{\pi}(B), \qquad \pi^{*}(A \odot B) = \pi^{*}(A)\pi^{*}(B)$$

which explains why evaluations at 0 are unnecessary.

♪ For aff(2) we have

$$\begin{aligned} C_1 &= A_1 + B_1 e^{A_2} \\ C_2 &= A_2 + B_2 \end{aligned}$$

and the pi-matrices

$$\pi^{\ddagger}(A) = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}, \qquad \pi^{*}(A) = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

Calculating the Jacobians, we find

$$\frac{\partial C_i}{\partial A_j} = \begin{pmatrix} 1 & B_1 e^{A_2} \\ 0 & 1 \end{pmatrix}, \qquad \frac{\partial C_i}{\partial B_j} = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

We readily verify the corresponding relations.

5 Procedure

⋆ **Start with** a given basis for the Lie algebra.

⋆ **Find the coordinate map** via the characteristic equations for the left dual flow $\dot{A} = \alpha\pi^{\ddagger}(A)$.

With initial conditions $A(0) = A$, this yields $A(\alpha t) \odot A$.

Hence the map $\alpha \longrightarrow A$, by evaluating at $A = 0,\ t = 1$.

⋆ **Interpret** A as **momentum variables**

α as **canonical momenta**.

- ⋆ Dual variables are x to A, Y to α.

⋆ **Jacobians**

- ⋆ $\dfrac{\partial A}{\partial \alpha}$ expressed in terms of A is used for the **raising operators** Y.
- ⋆ $\dfrac{\partial \alpha}{\partial A}$ in terms of α computed as the algebraic inverse

is used to express the variables x in terms of abstract raising and lowering operators.

The x variables in that form are the **recursion operators**.

⋆ **Generic formulae**

$$Y \;=\; xW(D) \;=\; xU'(V(D))$$

$$x \;=\; YV'(D) \;=\; YU'(V)^{-1}$$

become

$$Y = x\,A'(\alpha(A))\Big|_{A\to D}\,, \qquad x = Y\,A'(\alpha)^{-1}\Big|_{\substack{Y\to R\\ \alpha\to V}}$$

⋆ **Canonical polynomials** $y_n(x) = Y^n 1$. Abstract raising and lowering operators on the basis y_n are

$$\begin{aligned} R_i\,y_n &= Y_i\,y_n = y_{n+\mathrm{e}_i} \\ V_i\,y_n &= n_i\,y_{n-\mathrm{e}_i} \end{aligned}$$

⋆ **Acting on the basis** y_n, x's yield recursion formulas. Basic expressions are (row vector times matrix) :

$$Y \;=\; x\,A'(\alpha(D))$$

$$x \;=\; R\,A'(V)^{-1}$$

⋆ **Including a canonical change-of-variables** in $\xi^{\ddagger}$ yields the general dvf

$$\hat{\xi}_i = x_\nu W_{\nu\lambda}(D)\pi^{\ddagger}_{i\lambda}(V(D))$$

with

$$e^{\alpha_\mu \hat{\xi}_\mu} 1 = e^{x_\mu U_\mu(A(\alpha))}$$

⋆ **In particular,**

$$\hat{\xi}_i = x_\nu A'(D)^{-1}_{\nu\lambda}\, \pi^{\ddagger}_{i\lambda}(A(D))$$

yields

$$e^{\alpha_\mu \hat{\xi}_\mu} 1 = e^{\alpha_\mu x_\mu}\ .$$

Now the coherent state is the same as for an abelian algebra, so we call these

ACS operators

6 Integral formula for the Jacobian

⋆ **To get** the canonical variables requires the Jacobian of the map $\alpha \longrightarrow A$.

Since one has the differential equations for A,

namely the characteristic equations $\dot{A} = \alpha\pi(A)$,

one would think it possible to find $\partial A/\partial\alpha$ directly in terms of the π-matrices.

This is the subject of an interesting theorem stated without proof.

⋆ Let $J = \partial A/\partial\alpha$ denote the Jacobian of the coordinate map $\alpha \longrightarrow A$. Then

$$J(\alpha) = \hat{\pi}(A(\alpha)) \int_0^1 \check{\pi}(A(s))\, ds$$

⋆ Alternatively, we have

$$J(\alpha) = \hat{\pi}^*(A(\alpha)) \int_0^1 \check{\pi}^{-1}(A(s))\, ds$$

⋆ For the raising operators, we want J as a function of A.

7 Canonical variables in the nonabelian case

⋆ **Canonical variables** can be combined with the Lie case. Let

$$Y = x_\nu \alpha_\mu \pi^{\ddagger}_{\mu\lambda}(V(D))W_{\nu\lambda}(D)$$

Then

$$e^{tY}e^{a\cdot x} = \exp\Big(x\cdot U\Big(A(t\alpha)\odot V(a)\Big)\Big)$$

Proof:

Acting on $e^{a\cdot x}$ we have

$$\begin{aligned} Ye^{a\cdot x} &= x_\nu \alpha_\mu \pi^{\ddagger}_{\mu\lambda}(V(a))W_{\nu\lambda}(a)e^{a\cdot x} \\ &= \alpha_\mu \pi^{\ddagger}_{\mu\lambda}(V(a))W_{\nu\lambda}(a)\,\frac{\partial}{\partial a_\nu}\,e^{a\cdot x} \end{aligned}$$

This latter is a vector field in the a-variables.

⋆ **The characteristic equations** are

$$\dot{a}_i = \alpha_\mu \pi^\ddagger_{\mu\lambda}(V(a))W_{i\lambda}(a)$$

⋆ **Multiplying by** $V'(a)$ yields

$$V'(a)_{k\lambda}\dot{a}_\lambda = \alpha_\mu \pi^\ddagger_{\mu k}(V(a))$$

⋆ **The left-hand side** is an exact derivative, $\dfrac{d}{dt}V(a(t))$. So these are characteristic equations for the left dual flow in the V-variables. Integrating, we have

$$V(a(t)) = A(t\alpha) \odot V(a)$$

In other words,

$$a(t) = U(A(t\alpha) \odot V(a))$$

⋆ **To the vector fields**

$$\xi_i^{\ddagger}(V(x)) = \pi_{i\lambda}^{\ddagger}(V(x))W_{\nu\lambda}(x)D_\nu$$

correspond the dvf's

$$\hat{\xi}_i = x_\nu W_{\nu\lambda}(D)\pi_{i\lambda}^{\ddagger}(V(D))$$

And with $\hat{X} = \alpha_\mu \hat{\xi}_\mu$,

$$e^{\hat{X}}1 = e^{x\cdot U(A(\alpha))}$$

Note that the $\hat{\xi}_i$ are the double dual in the canonical variables (Y, V).

⋆ **Now choose** U and A to be inverse maps, i.e., $V(z) = A(z)$. Then we have the nonabelian Lie algebra yielding the same result on the vacuum state, 1, as the abelian one, namely

$$\exp(\hat{X})1 = \exp(\alpha \cdot x)$$

8 Examples

♫ For HW we have the **coordinate map**

$$A_1 = \alpha_1, \qquad A_2 = \alpha_2 + \alpha_1\alpha_3/2, \qquad A_3 = \alpha_3$$

⋆ **The Jacobians** are

$$\frac{\partial A}{\partial \alpha} = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_3/2 & 1 & \alpha_1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\frac{\partial \alpha}{\partial A} = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_3/2 & 1 & -\alpha_1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

with the latter calculated as $\left(\frac{\partial A}{\partial \alpha}\right)^{-1}$.

⋆ In terms of A,

$$\frac{\partial A}{\partial \alpha}(A) = \begin{pmatrix} 1 & 0 & 0 \\ A_3/2 & 1 & A_1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

⋆ **Contracting with** x and replacing A by D yields the **raising operators**

$$Y_1 = x_1 + \tfrac{1}{2}x_2 D_3\,, \quad Y_2 = x_2\,, \quad Y_3 = x_3 + \tfrac{1}{2}x_2 D_1$$

⋆ These are commuting variables.

⋆ **The basic expansion** is

$$\begin{aligned} e^{\alpha_\mu Y_\mu}1 &= e^{\alpha_1 x_1} e^{x_2(\alpha_2+\alpha_1\alpha_3/2)} e^{\alpha_3 x_3} \\ &= \sum_{n\geq 0} \frac{\alpha^n}{n!}\, y_n(x) \end{aligned}$$

⋆ **Contracting with** R and replacing α by V in $\partial\alpha/\partial A$ yields the x-variables as **recursion operators**

$$x_1 = R_1 - R_2 V_3/2\,, \quad x_2 = R_2\,, \quad x_3 = -R_2 V_1/2 + R_3$$

⋆ **On the basis** y_n, we thus have

$$\begin{aligned} x_1\, y_n &= y_{n+\mathrm{e}_1} - \tfrac{1}{2}\, n_3\, y_{n+\mathrm{e}_2-\mathrm{e}_3} \\ x_2\, y_n &= y_{n+\mathrm{e}_2} \\ x_3\, y_n &= y_{n+\mathrm{e}_3} - \tfrac{1}{2}\, n_1\, y_{n-\mathrm{e}_1+\mathrm{e}_2} \end{aligned}$$

⋆ **Replacing** R by x, V by D and contracting with the transpose of $\pi^{\ddagger}(A(D))$, yields the **ACS representation** of the Lie algebra

$$\hat{\xi}_1 = x_1 - \tfrac{1}{2} x_2 D_3\,, \quad \hat{\xi}_2 = x_2\,, \quad \hat{\xi}_3 = x_3 + \tfrac{1}{2} x_2 D_1$$

which obey the commutation relations for the Heisenberg algebra while satisfying

$$\exp\,(\alpha_\mu \hat{\xi}_\mu) 1 = \exp \alpha_\mu x_\mu$$

$\boxed{♪}$ Aff

We have the coordinate map

$$A_1(\alpha) = \frac{\alpha_1}{\alpha_2}\left(e^{\alpha_2} - 1\right), \qquad A_2(\alpha) = \alpha_2$$

and

$$\pi^{\ddagger} = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}$$

The Jacobians are

$$\frac{\partial A}{\partial \alpha} = \begin{pmatrix} \alpha_1\left(e^{\alpha_2} - 1\right)/\alpha_2 & -\alpha_1\left(e^{\alpha_2} - 1 - \alpha_2\right)/\alpha_2^2 \\ 0 & 1 \end{pmatrix}$$

and

$$\frac{\partial \alpha}{\partial A} = \begin{pmatrix} \frac{\alpha_2}{e^{\alpha_2} - 1} & \frac{\alpha_1}{\alpha_2} - \alpha_1 \frac{1}{1 - e^{-\alpha_2}} \\ 0 & 1 \end{pmatrix}$$

from these the raising operators, recursion operators and ACS representation can be found as prescribed.

9 Concluding Remarks

⋆ **There are** many points for continued study. By specializing the coordinates one can find certain elements of the Lie algebra that generate classically interesting polynomials, such as Hermite polynomials via the Heisenberg algebra.

⋆ **The polynomials** found in the approach indicated here have particular structure depending on the Lie algebra. Exactly how these polynomials and the structure of the Lie algebra are related in some deeper way has not been clarified.

⋆ **Another source** of interest is, of course, the Jacobians. One can look at Jacobians of the form $\partial A(t)/\partial A(s)$, for $s < t$. As the Jacobians form a multiplicative family along paths, there are some possibilities for interesting dynamical systems, or perhaps, matrix-valued stochastic processes.

⋆ **Generally speaking,** it looks challenging and interesting to get some detailed information for classes of higher-dimensional Lie algebras. Certain classes of Lie algebras, such as symmetric Lie algebras, may allow for general structural results.