A Random Graph Proof for the Irreducible Case of the Markov Chain Tree Theorem

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1 Introduction

There are several proofs of the Markov chain tree theorem in the literature. Usually the proof proceeds in two steps. The first step proves the theorem for the special case of irreducible Markov chains. (This special case is interesting and useful in itself.) The second step uses the special case theorem as a lemma in the proof of the general case. The proofs in [3] and [1] proceed in that way, and those two references sketch some of the history of the problem.

In this paper we use a random graph approach to obtain a self contained and shorter proof of the special case. It uses no results from random graph theory (or indeed from anywhere else). We feel it is very simple and quite intuitive.[1]

This paper uses only terms, concepts, and notation that are needed to state and prove the special case. For example, even to state the general theorem requires the use of what the literature usually calls an "arborescence". But in the special case, an "arborescence" is obviously simply a spanning in-tree, so we need not even bother to define "arborescence".

2 The Theorem for Irreducible Chains

We think of the state diagram of a finite state Markov chain as a complete digraph, where the nodes correspond to the states and where associated with each edge is a non-negative transition probability. The states are numbered from 1 to n. Edge (i, j) is the edge from state i to state j. The real number p_{ij} is the transition probability associated with edge (i, j).

We call the chain irreducible if for any ordered pair of states there is a directed path of edges from the first state to the second along which all the transition probabilities are positive. Let P be the $n \times n$ matrix of transition probabilities with entries p_{ij}. (The matrix P is called irreducible if the chain is irreducible.)

A spanning in-tree is a rooted spanning tree in which all the edges point toward the root. An i-tree is a spanning in-tree whose root is node i (state i).

If t is a spanning in-tree, we define the number $\hat{t}$ to be the product of all the p_{ij} for which edge (i, j) is in t. We call $\hat{t}$ the value of the spanning in-tree t. For any node i, we define x_i to be the sum of the values of all the i-trees. Let $\bar{x}$ be the sum of the values of all the spanning in-trees, so $\bar{x} = \sum_i x_i$.

Let $\mathbf{x}$ be the row vector whose entries are the x_i's. We shall call $\mathbf{x}$ the *treevalues vector*. If the chain is irreducible then there is at least one spanning in-tree with non-zero value, so $\mathbf{x}$ is not the zero vector.

It is the purpose of this paper to present a proof of the following:

> **Theorem:** If P is the transition probabilities matrix of a irreducible finite state Markov chain, and $\mathbf{x}$ is the treevalues vector, then $\mathbf{x}P = \mathbf{x}$.

Note that the conclusion of the theorem is that $\mathbf{x}$ is an eigenvector of P with eigenvalue 1. It is well known that for an irreducible Markov chain, the subspace of eigenvectors with eigenvalue 1 is one dimensional.[2]

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[1] We are grateful to Mark Levene for pointing out some similarities between our proof and the proof in [4].

[2] From the Perron-Frobenius theorem [2, page 65] together with [2, page 100].