Matrix-Forest Theorems

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The Laplacian matrix of a graph G is $L(G)=D(G)-A(G)$, where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of vertex degrees. According to the Matrix-Tree Theorem, the number of spanning trees in G is equal to any cofactor of an entry of $L(G)$. A rooted forest is a union of disjoint rooted trees. We consider the matrix $W(G)=I+L(G)$ and prove that the (i,j)-cofactor of $W(G)$ is equal to the number of spanning rooted forests of G, in which the vertices i and j belong to the same tree rooted at i. The determinant of $W(G)$ equals the total number of spanning rooted forests, therefore the (i,j)-entry of the matrix $W^{-1}(G)$ can be considered as a measure of relative "forest-accessibility" of the vertex i from j (or j from i). These results follow from somewhat more general theorems we prove, which concern weighted multigraphs. The analogous theorems for (multi)digraphs are established. These results provide a graph-theoretic interpretation to the adjugate of the Laplacian characteristic matrix.

1. INTRODUCTION

Let G be a labeled graph on n vertices with adjacency matrix $A(G)=(a_{ij})$. The Laplacian (the Kirchhoff or the admittance) matrix of G is the n-by-n matrix $L(G)=(\ell_{ij})$ with $\ell_{ij}=-a_{ij}$ $(j\ne i,\;\; i,j=1,\ldots,n)$ and $\ell_{ii}=\sum_{j\ne i}a_{ij}=-\sum_{j\ne i}\ell_{ij}$ $(i=1,\ldots,n)$. According to the Matrix-Tree Theorem attributed to Kirchhoff (for its history, see [19]), any cofactor of an entry of $L(G)$ is equal to the number of spanning trees of G. Tutte (see [26]) has generalized this theorem to weighted multigraphs and multidigraphs. Bapat and Constantine [1] presented a version for graphs in which each edge is assigned a color. Merris [17] proposed an "edge version" of the Matrix-Tree Theorem and Moon [20] generalized it. Forman [9] considered the Kirchhoff theorem in a more general context of vector fields.

Another trend of literature studies the characteristic polynomial and the spectrum of the Laplacian matrix. For review of this literature we refer to [10, 11, 18]. We would like to mention here the research by Kelmans, who had published in 1965–1967 a series of results on the Laplacian characteristic polynomial and spectrum (see [13, 14], and the references therein), some of which were rediscovered later by other writers.

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In [14] Kelmans and Chelnokov have shown that the coefficients of the Laplacian characteristic polynomial can be represented through the numbers of spanning forests of G with fixed numbers of components. This result is closely related to the theorems in this paper and we use it in our proofs. Another close result has been obtained by Fiedler and Sedláček [8] (see Lemma 3 in the Appendix) and generalized in [5, 2, 21].

We consider the matrix $W(G) = I + L(G) = -Z(-1, G)$, where $Z(\lambda, G) = \lambda I - L(G)$ is the Laplacian characteristic matrix of G and I is the identity matrix. It turns out that $W(G)$ can be used for counting spanning rooted forests of G (a rooted forest is a union of disjoint rooted trees) somewhat similarly to as $L(G)$ can be used to count spanning trees. Namely, the determinant of $W(G)$ is equal to the number of all spanning rooted forests of G, and the cofactor $W^{ij}(G)$ is equal to the number of those spanning rooted forests, such that i and j belong to the same tree rooted in i. This is a simple consequence of Theorems 5 and 6 in Section 3.

Theorems 3 and 4 consider an arbitrary multidigraphs Γ and provide an analogous relation between the Kirchhoff matrix of Γ and the numbers of spanning *diverging* forests of Γ. These results allow us to consider the matrices $W^{-1}(G)$ and $W^{-1}(\Gamma)$ as matrices of *relative forest-accessibilities* in the multigraph G and the multidigraph Γ.

It can be interesting to compare Theorems 3–7 with the corresponding results on the adjacency characteristic matrix (see [6, Subsections 1.9.1 and 1.9.5] or the original articles by Kasteleyn and Ponstein cited therein, and also [23]). About counting forests see [7, 12, 19]. Liu and Chow [15] obtained a rather complicated expression for the number of k-component spanning forests of a graph through the principal minors of the Laplacian matrix. Myrvold [22] gave a simpler graph-theoretic proof of some version of their result and discovered a polynomial algorithm for calculating this number of k-component spanning forests. The ideas of her proof are similar to those of Kelmans and Chelnokov [14].

In the following section, we give some graph-theoretic notation and statements of the Matrix-Tree Theorem for weighted multigraphs and multidigraphs, which will help us to formulate and prove our results.

2. PRELIMINARIES

Let us remind some necessary graph-theoretic notions. A *subgraph* of a multigraph G is a multigraph all of whose vertices and edges belong to the vertex and edge sets of G. A *spanning subgraph* of G is a subgraph of G whose vertex set coincides with the vertex set of G. A *forest* is a cycleless graph. A *tree* is a connected forest. A *rooted tree* is a tree with one marked vertex called a *root*. Formally, the rooted tree is a pair (T, r), where T is the tree and r is its vertex. A *component* of a multigraph G is any maximal (by inclusion) connected subgraph of G. A *rooted forest* can be defined as a forest with one marked vertex in each component. Obviously, a rooted forest is a union of disjoint rooted trees.

A *path* from vertex i to vertex j in a multidigraph Γ is an alternating sequence of distinct vertices and arcs having each arc directed from the previous vertex to the next one; i is the first vertex, j is the last one. Sometimes we will consider a path as a subgraph of Γ. A digraph is called a *directed tree* (a *directed forest*) if the graph obtained from it by replacement of all its arcs by edges is a tree (a forest). The definitions for *directed rooted tree* and *directed rooted forest* are analogous to the definitions of rooted tree and rooted forest (we will omit the word "directed" while talking about subgraphs of Γ). A *diverging tree* is a directed rooted tree, containing paths from the root to all other vertices. A *diverging forest* is a directed rooted forest, all whose components are diverging trees.

The Matrix-Tree Theorem asserts that for any graph G, all cofactors of entries of $L(G)$ are equal to each other and their common value is the number of spanning trees in G.

Now suppose G is a weighted multigraph with vertex set $V(G) = \{1, \ldots, n\}$ and suppose ε_{ij}^m is the weight of the mth edge between vertices i and j in G. This weight will be also referred to as a *conductance* of the edge. However, we will not forbid ε_{ij}^m to be negative. The Kirchhoff matrix of G is the n-by-n matrix $L = L(G) = (\ell_{ij})$ with $\ell_{ij} = -\sum_{m=1}^{a_{ij}} \varepsilon_{ij}^m \quad (j \neq i, \quad i, j = 1, \ldots, n)$ and $\ell_{ii} = -\sum_{j \neq i} \ell_{ij} \quad (i = 1, \ldots, n)$, where a_{ij} is the number of edges between i and j. Denote by L^{ij} the cofactor of ℓ_{ij} in L. The product of the conductances of all edges belonging to a subgraph H of the multigraph G will be referred to as the *weight* or *transmission coefficient* of H and denoted by $\varepsilon(H)$. The weight of a subgraph without edges is assumed to be 1. For every nonempty set of subgraphs $\mathcal{G}$, its weight is defined as follows: $\varepsilon(\mathcal{G}) = \sum_{H \in \mathcal{G}} \varepsilon(H)$. Set the weight of the empty set to be zero. Let $\mathcal{T}(G) = \mathcal{T}$ be the set of all spanning trees of G.

Tutte's [26] generalization of the Matrix-Tree Theorem can be formulated as follows.

THEOREM 1 (Matrix-Tree Theorem for weighted multigraphs) *For any weighted multigraph G and for any $i, j \in V(G)$, $L^{ij} = \varepsilon(\mathcal{T})$.*

Tutte [26] also developed a parallel theory for multidigraphs.

Let Γ be a multidigraph with vertex set $V(\Gamma) = \{1, \ldots, n\}$ and suppose ε_{ij}^m is the weight (or the conductance) of the mth arc from i to j in Γ. The Kirchhoff matrix $L(\Gamma)$ of Γ is the n-by-n matrix $L = L(\Gamma) = (\ell_{ij})$ with $\ell_{ij} = -\sum_{m=1}^{a_{ji}} \varepsilon_{ji}^m \ (j \neq i, \ \ i, j = 1, \ldots, n)$ and $\ell_{ii} = -\sum_{j \neq i} \ell_{ij} \ (i = 1, \ldots, n)$, where a_{ji} is the number of arcs from j to i in Γ. Notice that ℓ_{ii} is the total conductance of the arcs *converging* to i. The definition for weight of a subgraph of Γ is analogous to the corresponding definition for multigraphs. Suppose $\mathcal{T}^i$ is the set of spanning trees of Γ diverging from i.

THEOREM 2 (Matrix-Tree Theorem for weighted multidigraphs) *For any weighted multidigraph* Γ *and for any* $i, j \in V(\Gamma)$, $L^{ij} = \varepsilon(\mathcal{T}^i)$.

Note that in the directed case entries in different rows of L may have different cofactors, but all the entries of one row have equal cofactors.

For simplicity, Tutte formulates this theorem (as well as the previous one) only for diagonal cofactors L^{ii}. The "directed" Matrix-Tree Theorem concerning arbitrary L^{ij} is given in Harary and Palmer [12].

In the following section, we give somewhat analogous theorems on spanning converging forests of a multidigraph Γ and on spanning rooted forests of a multigraph G.

3. MATRIX-FOREST THEOREMS

Consider the matrices $W(\Gamma) = I + L(\Gamma)$ and $W(G) = I + L(G)$. $W^{ij}(\Gamma)$ and $W^{ij}(G)$ will denote the cofactors of the (i, j)-entries of $W(\Gamma)$ and $W(G)$.

Suppose $\mathcal{F}(\Gamma) = \mathcal{F}$ is the set of all spanning diverging forests of Γ and $\mathcal{F}^{i\to j}(\Gamma) = \mathcal{F}^{i\to j}$ is the set of those spanning diverging forests of Γ, such that i and j belong to the same tree diverging from i. Let $W = W(\Gamma)$, $W^{ij} = W^{ij}(\Gamma)$.

THEOREM 3 *For any weighted multidigraph* Γ, $\det W = \varepsilon(\mathcal{F})$.

THEOREM 4 *For any weighted multidigraph* Γ *and for any* $i, j \in V(\Gamma)$, $W^{ij} = \varepsilon(\mathcal{F}^{i\to j})$.

As usual, these theorems have dual counterparts concerning *converging* forests. Theorems 3 and 4 can be derived in the shortest way from one version of Chaiken's result [2], namely, by putting $U = W = \varnothing$ and then $U = \{i\}$, $W = \{j\}$ in the first formula in page 328 (cf. [21, Theorem 3.1]). In the Appendix of this paper, we give another proof which demonstrates some interesting relations of Matrix-Forest Theorems to the results in [8, 13, 14, 16].

Suppose $\mathcal{F}(G) = \mathcal{F}$ is the set of all spanning rooted forests of a weighted multigraph G and $\mathcal{F}^{ij}(G) = \mathcal{F}^{ij}$ is the set of those spanning rooted forests of G, such that i and j belong to the same tree rooted in i. Let $W = W(G)$, $W^{ij} = W^{ij}(G)$.

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THEOREM 6 *For any weighted multigraph* G *and for any* $i, j \in V(G)$, $W^{ij} = \varepsilon(\mathcal{F}^{ij})$.

As the matrix W of a weighted multigraph is symmetrical, Theorem 6 remains true if we replace $\mathcal{F}^{ij}$ by $\mathcal{F}^{ji}$ in the right-hand side. In the Appendix, Theorems 5 and 6 are derived from Theorems 3 and 4.

If the weights ε_{ij}^m are non-negative, then by Theorems 3 and 5, the matrix W is non-singular. If the matrix W^{-1} exists, we will denote it by $Q = \left(q_{ij}\right)$ (both for a weighted multidigraph Γ and for a weighted multigraph G). Then $Q = (\det W)^{-1}W^*$, where $W^* = (W^{ij})^\intercal$ is the adjugate of W. Theorems 3-6 imply the following main theorem.

THEOREM 7 *1. For any weighted multidigraph Γ, if the matrix $Q = W^{-1}$ exists, then* $q_{ij} = \varepsilon(\mathcal{F}^{j\to i})\big/\varepsilon(\mathcal{F}),\;\; i,j = 1,\ldots,n.$
2. For any weighted multigraph G if the matrix $Q = W^{-1}$ exists, then
$q_{ij} = \varepsilon(\mathcal{F}^{ji})\big/\varepsilon(\mathcal{F}),\;\; i,j = 1,\ldots,n.$

It can be seen that $\sum_{j=1}^{n} q_{ij} = 1$ $(i = 1,\ldots,n)$ both for directed and undirected weighted multigraphs. This follows, for example, from the facts that for any $i \in V(\Gamma)$, the sets $\mathcal{F}^{j\to i}$ $(j = 1,\ldots,n)$ are non-overlapping and $\bigcup_{j=1}^{n} \mathcal{F}^{j\to i} = \mathcal{F}$ (respectively, for any $i \in V(G)$, the sets $\mathcal{F}^{ji}$ $(j = 1,\ldots,n)$ are non-overlapping and $\bigcup_{j=1}^{n} \mathcal{F}^{ji} = \mathcal{F}$).

If the weights of all arcs of Γ (of all edges of G) are ones, Theorems 3–7 tell us about the *numbers* of corresponding spanning forests (which are equal to their summary TC's in this case).

Theorem 7 allows us to consider the matrix $Q = W^{-1}$ as the matrix of *relative forest-accessibilities* of the vertices of Γ (or G).

Theorems 3 and 4 were formulated in [3] and Theorems 5 and 6 in [24]. The latter results were used in [25] for the analysis of one method of preference aggregation. That paper implicitly contains proofs of these theorems (in the case of equal weights ε_{ij}^m), different from the proofs given here. In [4] we analyze the properties of relative forest-accessibilities and exploit them to introduce a new family of sociometric indices.

Theorems 5–7 were used in [25] for the analysis of one method of preference aggregation. That paper contains implicitly the proofs of these theorems (in the case of equal weights ε_{ij}^m), different from the proofs given here.

Remark Lemma 5 in the Appendix provides an interpretation for the adjugate of the characteristic matrix of $-L(\Gamma)$. Replacing $\varepsilon(F)$ by $(-1)^{d(F)}\varepsilon(F)$ in (2) ($d(F)$ is the number of arcs in F), we obtain a representation for the adjugate of the characteristic matrix of $L(\Gamma)$. To get analogous representations in the undirected case, it suffices to replace $\mathcal{F}_{\varphi\cup\{i\}}^{i\to j}$ by $\mathcal{F}_{\varphi\cup\{i\}}^{ij}$ in (2) ($\mathcal{F}_{\varphi\cup\{i\}}^{ij} = \mathcal{F}^{ij}\cap\mathcal{F}_{\varphi\cup\{i\}}$, and $\mathcal{F}_{\varphi\cup\{i\}}$ is the set of spanning rooted forests of G, having $|\varphi\cup\{i\}|$ components rooted in the vertices of $\varphi\cup\{i\}$). The latter is obvious by the argument used in the proof of Theorems 5 and 6.

APPENDIX

Prior to proving Theorems 3–6 we introduce some additional notation and prove several lemmas.

$p(\lambda)$ is the characteristic polynomial of the matrix $-L = -L(\Gamma)$;

$W_\lambda = \lambda I + L(\Gamma)$ (λ is a real number);

$E = E(\Gamma)$ is the arc set of Γ;

φ is a subset of $V = V(\Gamma) = \{1, \ldots, n\}$;

$L_{-\varphi}(\Gamma) = L_{-\varphi}$ is the matrix obtained from $L(\Gamma)$ by deleting the rows and columns corresponding to the vertices of φ; we will use the analogous expression $U_{-\psi}$ for an arbitrary n-by-n matrix U and $\psi \subseteq \{1, \ldots, n\}$.

Γ_φ is the weighted multidigraph obtained from Γ by identifying all the vertices of φ;

φ^* is the vertex of Γ_φ being a result of this identification; any arc incident to some vertex of φ in Γ have the corresponding arc incident to φ^* in Γ_φ;

"Γ-tree" is a spanning diverging tree of Γ;

"Γ-forest" is a spanning diverging forest of Γ;

$\mathcal{T}_{\varphi^*}$ is the set of Γ_φ-trees diverging from φ^*; if $\varphi = \varnothing$, we set $\mathcal{T}_{\varphi^*} = \varnothing$;

$\mathcal{F}_\varphi$ is the set of Γ-forests with $|\varphi|$ components that diverge from the vertices of φ;

$\mathcal{F}_\varphi^{j \to i} = \mathcal{F}^{j \to i} \bigcap \mathcal{F}_\varphi$ (if $j \notin \varphi$, then $\mathcal{F}_\varphi^{j \to i} = \varnothing$);

Lemma 1 *Let Γ_1 and Γ_2 be weighted multidigraphs with the same vertex set. Suppose that the arc set of Γ_2 can be obtained from that of Γ_1 by replacing some arc (i,j) (with some weight ε_{ij}) by two arcs from i to j with the weights ε'_{ij} and ε''_{ij} such that $\varepsilon'_{ij} + \varepsilon''_{ij} = \varepsilon_{ij}$. Then*

(i) $W(\Gamma_1) = W(\Gamma_2)$;

(ii) for any vertices α and β, the value $\varepsilon(\mathcal{F}^{\alpha \to \beta})$ is the same for Γ_1 and Γ_2.

Proof (*i*) is obvious. (*ii*) holds since for any $F \in \mathcal{F}^{\alpha \to \beta}(\Gamma_1)$ there are two corresponding forests in $\mathcal{F}^{\alpha \to \beta}(\Gamma_2)$ with the same summary weight. ∎

Based on Lemma 1, we conclude that it suffices to prove Theorems 3 and 4 only for weighted *digraphs*. Thus, we will assume that Γ has no multiple arcs.

The following three lemmas are directed (and weighted) counterparts of certain results of Kelmans [13] and Kelmans and Chelnokov [14].

Lemma 2 *Let $\varphi \subseteq V$. Then, in terms of the notation above,* $\det L_{-\varphi} = \varepsilon(\mathcal{T}_{\varphi^*})$.

Proof If $\varphi = \varnothing$, we have zero in both sides of the equality. For $\varphi \neq \varnothing$, let $L(\Gamma_\varphi)$ be the Kirchhoff matrix of Γ_φ. Suppose $L_{-\{\varphi^*\}}(\Gamma_\varphi) = L_{-\varphi^*}(\Gamma_\varphi)$ is the matrix obtained from $L(\Gamma_\varphi)$ by deleting the row and column corresponding to φ^*. Then the desired equality is valid since by Theorem 2, $\det L_{-\varphi^*}(\Gamma_\varphi) = \varepsilon(\mathcal{T}_{\varphi^*})$, and $L_{-\varphi^*}(\Gamma_\varphi) = L_{-\varphi}$. ∎

Lemma 3 (Fiedler and Sedláček [8], cf. [1, 9]) *For any* $\varphi \subseteq V$, $\det L_{-\varphi} = \varepsilon(\mathcal{F}_\varphi)$.

We are proving Lemma 3 here, since this proof is very short.

Proof By Lemma 2, it suffices to prove the equality $\varepsilon(\mathcal{F}_\varphi) = \varepsilon(\mathcal{T}_{\varphi^*})$, which holds for any $\varphi \neq \varnothing$ since identifying the vertices of φ transforms any Γ-forest belonging to $\mathcal{F}_\varphi$ into a Γ_φ-tree diverging from φ^*, and this correspondence is one-to-one. If $\varphi = \varnothing$, we have zero in both sides. ■

Let $p(\lambda) = \det(\lambda I + L) = \sum\limits_{k=0}^{n} c_k \lambda^k$ be the characteristic polynomial of $-L$.

Lemma 4 $c_k = \sum\limits_{\substack{\varphi\subseteq V\\ |\varphi|=k}} \varepsilon(\mathcal{F}_\varphi), \quad k = 0, \ldots, n.$

Proof In view of Lemma 3, this follows from the fact that c_k is equal to the sum of degree k principal minors of L. ■

Proof of Theorem 3 Using Lemma 4, we have

$$\det W = \det(I + L) = p(1) = \sum_{k=0}^{n} \sum_{\substack{\varphi\subseteq V\\ |\varphi|=k}} \varepsilon(\mathcal{F}_\varphi) = \sum_{\varphi\subseteq V} \varepsilon(\mathcal{F}_\varphi) = \varepsilon(\mathcal{F}). \quad ■$$

Suppose $W_\lambda = \lambda I + L$ and

$$W_\lambda^{ij} = \sum_{k=0}^{n-1} b_k \lambda^k, \qquad i, j \in V \tag{1}$$

is the cofactor of the (i, j)-entry of W_λ.

Lemma 5 *In terms of the notation above,*

$$b_k = \sum_{\substack{\varphi\subseteq V\smallsetminus\{i,j\}\\ |\varphi|=k}} \varepsilon(\mathcal{F}^{i\to j}_{\varphi\cup\{i\}}), \qquad k = 0, \ldots, n-1. \tag{2}$$

Proof It is easy to see that

$$b_k = \sum_{\substack{\varphi\subseteq V\smallsetminus\{i,j\}\\ |\varphi|=k}} L^{ij}_{-\varphi} \quad (k = 0, \ldots, n-1), \tag{3}$$

where $L^{ij}_{-\varphi}$ is the cofactor in the matrix $L_{-\varphi}$ of the (i, j)-entry of L.

1. $i \neq j$. To obtain an expression for $L^{ij}_{-\varphi}$, we will use a theorem by Maybee (see [16]), which can be formulated as follows. For any n-by-n matrix $U = (u_{ij})$, the representation of a cofactor U^{ij},

$$U^{ij} = \sum_k \varepsilon(P_k^{i\to j}) \det U_{-\psi_k}, \tag{4}$$

is valid for $i \neq j$, where $P_k^{i\to j}$ is the kth path from i to j in an arbitrary weighted digraph $\Gamma(U)$ (with vertex set $\{1, \ldots, n\}$) connected with U by the following relations:
– if $i \neq j$ and $u_{ij} \neq 0$, then the arc (j, i) belongs to arc set $E(\Gamma(U))$ and has the weight $(-u_{ij})$;
– if $i \neq j$ and $u_{ij} = 0$, then the arc (j, i) has zero weight or $(j, i) \notin E(\Gamma(U))$.
ψ_k in (4) denotes the set of the vertices entering $P_k^{i\to j}$.

Notice that matrix L and weighted digraph Γ satisfy these conditions (recall that by our assumption, Γ has no multiple arcs). Therefore, these conditions are obeyed for $L_{-\varphi}$ and the subgraph of Γ induced on the vertex subset $V \smallsetminus \varphi$. Hence, by (4) and Lemma 3, we have

$$L^{ij}_{-\varphi} = \sum_k \varepsilon(P_k^{i\to j}) \det L_{-(\varphi\cup\psi_k)} = \sum_k \varepsilon(P_k^{i\to j}) \varepsilon(\mathcal{F}_{\varphi\cup\psi_k}). \tag{5}$$

Note that if $F \in \mathcal{F}_{\varphi\cup\psi_k}$ then the union of $P_k^{i\to j}$ and F belongs to $\mathcal{F}^{i\to j}_{\varphi\cup\{i\}}$. On the other hand, any $F' \in \mathcal{F}^{i\to j}_{\varphi\cup\{i\}}$ can be uniquely decomposed into a union of certain $P_k^{i\to j}$ and $F \in \mathcal{F}_{\varphi\cup\psi_k}$. Therefore, (5) implies

$$L^{ij}_{-\varphi} = \varepsilon(\mathcal{F}^{i\to j}_{\varphi\cup\{i\}}). \tag{6}$$

2. $i = j$. Lemma 3 implies $L^{ij}_{-\varphi} = L_{-(\varphi\cup\{i\})} = \varepsilon(\mathcal{F}_{\varphi\cup\{i\}})$. Since $\mathcal{F}_{\varphi\cup\{i\}} = \mathcal{F}^{i\to i}_{\varphi\cup\{i\}}$, we have (6) as well.

Now (6) and (3) yield (2). ■

Proof of Theorem 4 It suffices to put $\lambda = 1$ in (1) and use Lemma 5:

$$W^{ij} = W^{ij}_1 = \sum_{k=0}^{n-1} b_k = \sum_{k=0}^{n-1} \sum_{\substack{\varphi\subseteq V\smallsetminus\{i,j\} \\ |\varphi|=k}} \varepsilon(\mathcal{F}^{i\to j}_{\varphi\cup\{i\}})$$

$$= \sum_{\varphi\subseteq V\smallsetminus\{i,j\}} \varepsilon(\mathcal{F}^{i\to j}_{\varphi\cup\{i\}}) = \varepsilon(\mathcal{F}^{i\to j}). \qquad ■$$

Proof of Theorems 5 and 6 Let G be an arbitrary weighted *graph* (by undirected counterpart of Lemma 1, we assume that G has no multiple edges). Replace every edge of G, having, say, a weight ε, by two opposite arcs with the weight ε. The weighted digraph we obtain has the same Kirchhoff matrix as G. The desired statements follow from the fact that there exists a natural one-to-one correspondence between rooted forests of G and diverging forests of Γ. ■

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