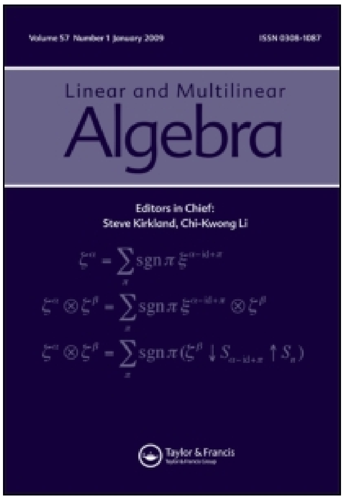

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Generalized Matrix Tree Theorem for Mixed Graphs

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In this article we provide a combinatorial description of an arbitrary minor of the Laplacian matrix (L) of a mixed graph (a graph with some oriented and some unoriented edges). This is a generalized Matrix Tree Theorem. We also characterize the nonsingular substructures of a mixed graph. The sign attached to a nonsingular substructure is described in terms of labeling and the number of unoriented edges included in certain paths. Nonsingular substructures may be viewed as generalized matchings, because in the case of disjoint vertex sets corresponding to the rows and columns of a minor of L, our generalized Matrix Tree Theorem provides a signed count over matchings between those vertex sets. A mixed graph is called quasi-bipartite if it does not contain a nonsingular cycle (a cycle containing an odd number of unoriented edges). We give several characterizations of quasi-bipartite graphs.

Keywords: Generalized matching; Laplacian; Matrix Tree Theorem; mixed graph; nonsingular substructure; quasi-bipartite graph; k-reduced spanning substructure

AMS Subject Classifications (1991): Primary: 05C50; Secondary: 15A15, 05C30, 05C70

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1. INTRODUCTION

The classical Matrix Tree Theorem in its simplest form [2, p. 219 and 4, p. 65] gives a combinatorial characterization of the adjoint of the Laplacian matrix of an oriented graph, in terms of spanning trees of the underlying graph. Since the adjoint of a matrix is just a special case of the compound of a matrix (the matrix consisting of fixed size minors of the matrix), and the Laplacian matrix of an oriented graph is a special case of the Laplacian matrix of a mixed graph (a graph with some oriented and some unoriented edges), it is reasonable to expect a much more general theorem, which could provide additional insight into a formula that otherwise may seem somewhat mysterious. An earlier paper [5] gives a combinatorial characterization of the trace of the compound of the Laplacian in an unoriented graph. To complete the process of generalizing the Matrix Tree Theorem, we show in this paper how to interpret each entry in the compound as a signed count of certain spanning-tree-like structures (called generalized matchings) in the mixed graph. The complete classical theorem now follows as a corollary, and we are also able to view certain entries as giving a signed count of matchings between disjoint subsets of the vertices in the graph. We discuss the edge version of the Laplacian of a mixed graph in [1]. For a general survey of results about the Laplacian, see [8].

Chaiken has discussed an All Minors Matrix Tree Theorem in [3]. His case for signed (undirected) graphs [3, p. 326] comes closest to our description. His nonsingular substructures are essentially the same as ours, except that they are made "spanning" by adding trees consisting of single vertices. Our approach can easily be modified to get results about matrices with entries representing weights given to all edges of the mixed graph. However, keeping those technicalities away has made our exposition easier, using only basic linear algebraic techniques and simple graph theoretic connections. Our method of determining the sign of each summand in a counting expression brings out more structural information about the nonsingular substructures.

After giving some preliminaries in the second section, we characterize nonsingular substructures required to describe off-diagonal minors of the Laplacian in the third section. In the fourth section, we provide the description of the sign attached with the nonsingular substructure. The last section focuses on characterizing quasi-bipartite graphs. It is clear that the incidence matrices of quasi-bipartite graphs are totally unimodular, and as such, can play an interesting role in applications to linear programming [6].

2. PRELIMINARIES

Let $G = (V, E)$ be a mixed graph with $\nu = \nu(G)$ vertices $V = V(G) = \{v_1, v_2, \ldots, v_\nu\}$ and $\varepsilon = \varepsilon(G)$ edges $E = E(G) = \{e_1, e_2, \ldots, e_\varepsilon\}$. Parallel edges and loops are permitted. In this paper we allow some of the edges to receive an orientation, thereby including both the classical approach of orienting all edges [2] and the unoriented approach [5] as extreme cases. Thus, in our graph G some of the edges have a specified head and tail, while others do not. It is important to stress, however, that our mixed graphs are considered undirected graphs in terms of defining paths, cycles, spanning trees, connectedness, *etc.* It turns out that oriented loops play no useful role in this theory, so we will assume that G has none of them.

The incidence matrix of G is the $\nu \times \varepsilon$ matrix $M = M(G) = [m_{ij}]$ whose entries are given by $m_{ij} = 1$ if e_j is an unoriented link (*i.e.*, nonloop) incident to v_i or if e_j is an oriented edge with head v_i, $m_{ij} = -1$ if e_j is an oriented edge with tail v_i, $m_{ij} = 2$ if e_j is a loop (necessarily unoriented) at v_i, and $m_{ij} = 0$ otherwise. Thus, every column of M contains either exactly two 1s, or exactly one 2, or exactly one 1 and one -1, with the other entries all 0. The *mixed Laplacian matrix* of G is defined as $L = L(G) = MM^t = [l_{ij}]$, where M^t denotes the transpose of M. It is easy to see that the diagonal entries of L give the degrees of the vertices with, however, each loop contributing 4 to the count, and the off-diagonal entry l_{ij} gives the number of unoriented edges joining v_i and v_j minus the number of oriented edges joining them (in either direction). For a matrix M, given subsets S and T of the sets of rows and columns respectively, $M[S, T]$ denotes the submatrix of M consisting of rows from S and columns from T.

We need to consider the *substructures* of G in which we may have deleted some edges or some vertices, or possibly both. We allow the possibility of deleting vertices without deleting the edges incident to them, although we will assume that each undeleted edge is incident to at least one undeleted vertex. Notions of connectivity and component are applied to substructures in the usual way; in particular, paths consist of alternating sequences of vertices and incident edges (with orientation ignored). Each substructure R of G gives rise to a submatrix $M(R)$ of $M(G)$ in the obvious way, using the $\nu(R)$ rows corresponding to $V(R)$ (the vertices in R), and the $\varepsilon(R)$ columns corresponding to $E(R)$ (the edges in R). We call the substructure R *nonsingular* if $M(R)$ is square and has nonzero determinant; otherwise, we call R *singular*. For example, if we take a spanning tree of a connected graph on ν vertices and delete one vertex (but not the edges incident to that vertex), then the resulting structure, which we call a *rootless spanning tree*, has $\nu - 1$ vertices and $\nu - 1$ edges. It is easy to see by looking at the submatrix that this is a nonsingular substructure of the original connected graph, and indeed, the determinant of the submatrix is ± 1. Note that a rootless spanning tree will be nonconnected if the deleted vertex had degree greater than 1 in the tree. The following result describes the corresponding situation for a cycle in a mixed graph.

LEMMA 1 *Let G be a mixed graph which is a cycle on ν vertices. Then the cycle is nonsingular if and only if it contains an odd number of unoriented edges. Furthermore, in that case, $det(M(G)) = \pm 2$.*

Proof If the cycle is a loop then the loop must be unoriented and hence has exactly one unoriented edge. Also, in this case, M is a 1×1 matrix with 2 as the entry and so its determinant is 2. So we assume that $\nu \geq 2$. We may assume, after a relabeling of the vertices if necessary, that the nonzero entries of $M(G)$ occur precisely at positions (i, i), $(i+1, i)$, for $i = 1, 2, \ldots, \nu - 1$, and $(1, \nu)$ and (ν, ν), and that the $(1, 1)$th entry is 1. Then expanding along the top row, we see that the determinant of M is, up to sign, given by $1 + (-1)^{p+\nu+1}$ if the $(1, \nu)$th entry is 1, and $1 + (-1)^{p+\nu}$ if the $(1, \nu)$th entry is -1, where p is the number of -1s in all rows of M other than the first one. Thus, M is nonsingular if and only if the sum of the number of vertices in G and the number of oriented edges in G is odd, *i.e.*, if and only if the number of unoriented edges in G is odd. It clearly follows from the discussion that $\det(M(G)) = \pm 2$ in this case. ■

A connected mixed graph containing exactly one cycle, with that cycle being nonsingular, is called a *nonsingular unicyclic graph*. Thus, a nonsingular unicyclic graph consists of a nonsingular cycle (possibly a loop) together with (possibly trivial) trees growing out of each vertex in the cycle. A mixed graph G will be called *quasi-bipartite* if it does not contain a nonsingular cycle. Thus, a mixed graph with all edges unoriented is quasi-bipartite if and only if it is bipartite; and a mixed graph with all edges oriented is always quasi-bipartite.

Given a mixed graph G, we let $\omega = \omega(G)$ and $\omega_0 = \omega_0(G)$ denote the number of components and the number of quasi-bipartite components of G, respectively. We set $\omega_1 = \omega_1(G) = \omega - \omega_0$. We can now prove the basic structure lemma.

LEMMA 2 *Let R be a substructure of a mixed graph with $\nu(R) = \varepsilon(R)$. Then unless every component of R has an equal number of vertices and edges, $det(M(R)) = 0$. If every component of R has an equal number of vertices and edges, then every component of R is a unicyclic graph or a rootless tree. If any one of the components is a singular unicyclic graph, then $det(M(R)) = 0$; otherwise, $det(M(R)) = \pm 2^{\omega_1(R)}$.*

Proof The first claim follows from the Laplace expansion of the determinant [7, p.14]. Thus, suppose that every component of R has an equal number of vertices and edges. Then by a permutation of the rows and columns we can put $M(R)$ in block diagonal form with square blocks, corresponding to the components of R. We claim that each component of R either is a rootless tree or consists of a cycle with (possibly trivial) rooted trees growing out of the vertices of the cycle. Indeed, if such a component has a vertex incident to exactly one edge in R, then we can remove the vertex and this edge and proceed inductively; otherwise every vertex must have degree exactly 2, and hence R is a cycle. The determinant of $M(R)$ can be evaluated by taking the product of the determinants of the diagonal blocks. If a component is a rootless tree then the determinant of the corresponding block is ± 1. If a component is unicyclic, then by Lemma 1, the determinant of the corresponding block is 0 or ± 2 according as whether the cycle in the component is singular or not. Hence, the result is proved. ■

We call a subgraph S of a connected mixed graph G an *essential spanning subgraph* of G if either G is quasi-bipartite and S is a spanning tree of G, or else G is not quasi-bipartite, $\nu(S) = \nu(G)$, and every component of S is a nonsingular unicyclic graph. Note in particular that an essential spanning subgraph of a connected mixed graph may be nonconnected, but each of its components H satisfies $\nu(H) = \varepsilon(H)$. An essential spanning subgraph of a nonconnected mixed graph is defined to be the union of one essential spanning subgraph from each component. It is easy to see that an essential spanning subgraph of G must contain $\nu - \omega_0$ edges.

Further, a *k-reduced spanning substructure* of a mixed graph G on ν vertices is a substructure of G containing $\nu - k$ vertices, each component of which contains an equal number of vertices and edges and has no singular cycles. It is easy to see that any k-reduced spanning substructure R of G has rootless trees and nonsingular unicyclic graphs as its components and satisfies $\nu(R) = \varepsilon(R) = \nu(G) - k$.

Every graph with at most k quasi-bipartite components has a k-reduced spanning substructure: simply take a spanning tree in k components (including all the quasi-bipartite ones) with one vertex deleted, together with a spanning nonsingular unicyclic subgraph in the remaining components. It is clear from the definitions that the ω_0-reduced spanning substructures of a mixed graph G (with ω_0 quasi-bipartite components) are in one-to-one correspondence with the essential spanning subgraphs of G with one vertex deleted from each quasi-bipartite component.

Now we are in a position to see that the rank of $M(G)$ is $\nu(G) - \omega_0(G)$. This immediately follows from observing that for a connected mixed graph G, the rank of $M(G)$ is $\nu(G) - 1$ if G is quasi-bipartite and $\nu(G)$ otherwise.

We also get the Principal Minor Version of the Matrix Tree Theorem for a mixed graph.

THEOREM 1 *Let G be a mixed graph, k a nonnegative integer not exceeding $\nu(G)$, and V_1 a subset of V containing $\nu - k$ vertices. Then*

$$det(L[V_1, V_1]) = \sum_R 4^{\omega_1(R)},$$

where the sum is taken over all k-reduced substructures R of G with $V(R) = V_1$.

Proof Since $L = MM^t$, by the Cauchy-Binet Theorem (see [7, p. 14]), we know that $\det(L[V_1, V_1])$ is the sum of the squares of the determinants of the square submatrices $M[V_1, E_1]$ where E_1 is a set of $\nu - k$ edges of G. By Lemma 2, the only nonzero contributions to this sum come from substructures R of G each component of which is a rootless tree or a nonsingular unicyclic graph, and the contribution is $(\pm 2)^2 = 4$ for each nonsingular cycle in the substructure. That gives the right-hand side of the equation. ■

For the following corollary we need to recall that the rth *compound* of an $m \times n$ matrix A is the $\binom{m}{r} \times \binom{n}{r}$ matrix $C_r(A)$ whose (i,j)th entry is the determinant of the matrix obtained from A by using the rows of the ith r-subset of the set of all rows of A, and the columns of the jth r-subset of the set of all columns of A.

COROLLARY 1 *Let G be a mixed graph. Then*

$$trace(C_{\nu-\omega_0}(L)) = \sum_S r(S)4^{\omega_1(S)},$$

where the sum is taken over all essential spanning subgraphs S of G, and r(S) is the product of the numbers of vertices in the quasi-bipartite (tree) components of S.

COROLLARY 2 *Let G be a mixed graph with no quasi-bipartite component. Then*

$$det(L(G)) = \sum_R 4^{\omega_1(R)},$$

where the sum is taken over all essential spanning subgraphs of G.

3. OFF-DIAGONAL MINORS

In this section, we provide a characterization of the nonsingular substructures needed to describe the off-diagonal minors of the Laplacian of the mixed graph G.

In order to study the off-diagonal minors of the Laplacian of G, we need to relativize the notion of nonsingularity. If $V \subseteq V(G)$ and $E \subseteq E(G)$, then we denote the substructure of G consisting of the vertices in V and the edges in E by $\mathcal{S}(V, E)$ and we denote its incidence matrix by $M[V, E]$, using rows of $M(G)$ corresponding to V and columns corresponding to E. Recall that $\mathcal{S}(V,E)$ is nonsingular if $|V| = |E|$ and $\det(M[V,E]) \neq 0$; and this is the case if and only if every component of $\mathcal{S}(V,E)$ is either a rootless tree or a nonsingular unicyclic subgraph.

Suppose that V_1 and V_2 are two subsets of $V(G)$, each having cardinality r, where $1 \leq r \leq \nu$, and suppose that $E \subseteq E(G)$ with $|E| = r$. We define $\mathcal{S}(V_1 \cup V_2, E)$ to be *nonsingular relative to* V_1 *and* V_2 if $\mathcal{S}(V_1, E)$ and $\mathcal{S}(V_2, E)$ are both nonsingular. This definition generalizes our earlier one, of course, if $|V_1 \cap V_2| = |E|$, since in that case necessarily $V_1 \cap V_2 = V_1 = V_2$. It is easy to see that if $\mathcal{S}(V_1 \cap V_2, E)$ is nonsingular relative to V_1 and V_2, then each edge in E has to contain at least one vertex in V_1 and at least one vertex in V_2. Thus, an edge in E could have both endpoints in $V_1 \cap V_2$, one endpoint in $V_1 \cap V_2$ and the other end dangling, one endpoint in $V_1 \cap V_2$ and the other endpoint in just one of the two vertex sets, or one endpoint in $V_1 \backslash V_2$ and the other in $V_2 \backslash V_1$.

The following theorem characterizes the nonsingular substructures of this kind.

THEOREM 2 *Let G be a mixed graph. Let V_1, $V_2 \subseteq V(G)$ and $E \subseteq E(G)$ with $|V_1| = |V_2| = |E|$. Then $\mathcal{S}(V_1 \cup V_2, E)$ is nonsingular relative to V_1 and V_2 if and only if each component is either a nonsingular substructure of $\mathcal{S}(V_1 \cap V_2, E)$ or a tree with exactly one vertex in each of $V_1 \backslash V_2$ and $V_2 \backslash V_1$.*

The sufficiency of the condition is clear. The proof of necessity follows from the following two lemmas about the components of $\mathcal{S}(V_1 \cup V_2, E)$. In these proofs, we use the observation that if a nonsingular $m \times m$ matrix has a zero submatrix of size $r \times s$, then necessarily $r + s \leq m$.

LEMMA 3 *Let $\mathcal{S}(V_1 \cup V_2, E)$ be nonsingular relative to V_1 and V_2. Suppose that K is a component of $\mathcal{S}(V_1 \cup V_2, E)$ and is also a component of $\mathcal{S}(V_1, E)$ or $\mathcal{S}(V_2, E)$. Then K is a component of $\mathcal{S}(V_1 \cap V_2, E)$.*

Proof Without loss of generality, let K be a component of $\mathcal{S}(V_1, E)$. It follows from the nonsingularity of $\mathcal{S}(V_1, E)$ that $|V(K)| = |E(K)|$. Let K have p vertices in $V_1 \cap V_2$ and q vertices in $V_1 \backslash V_2$. Then $p + q = |V(K)| = |E(K)|$. The submatrix of the nonsingular matrix $M(V_2, E)$ formed by the rows corresponding to $V_2 \backslash V(K)$ and the columns corresponding to $E(K)$ is the zero matrix, and so by the observation just preceding this lemma,

$$|V_2 \backslash V(K)| + |E(K)| \le |V_2|.$$

Thus,

$$(|V_2| - p) + (p + q) \le |V_2|,$$

and hence $q = 0$. Therefore, K is a component of $\mathcal{S}(V_1 \cap V_2, E)$. ∎

LEMMA 4 *Let $\mathcal{S}(V_1 \cup V_2, E)$ be nonsingular relative to V_1 and V_2. Suppose that K is a component of $\mathcal{S}(V_1 \cup V_2, E)$ but is a component of neither $\mathcal{S}(V_1, E)$ nor $\mathcal{S}(V_2, E)$. Then K is a tree containing exactly one vertex in each of $V_1 \backslash V_2$ and $V_2 \backslash V_1$.*

Proof Suppose that K contains t_1 vertices in $V_1 \backslash V_2$ and t_2 vertices in $V_2 \backslash V_1$. By our hypothesis, $t_1 \ge 1$ and $t_2 \ge 1$. The submatrix of the nonsingular matrix $M(V_1, E)$ formed by the rows corresponding to $V_1 \backslash V(K)$ and columns corresponding to $E(K)$ is the zero submatrix, and hence

$$|V_1 \backslash V(K)| + |E(K)| \le |V_1|.$$

It can be seen that $|V_1 \backslash V(K)| = |V_1| - |V(K)| + t_2$, and hence

$$1 \le t_2 \le |V(K)| - |E(K)| \le 1.$$

The last inequality follows since K is connected. Thus, these inequalities are in fact equalities. Similarly

$$t_1 = 1 = |V(K)| - |E(K)|.$$

Since $|V(K)| - |E(K)| = 1$, K is a tree and the proof is complete. ∎

4. GENERALIZED MATCHING AND SIGN

In this section, we present the generalized Matrix Tree Theorem. We first determine the sign attached to each nonsingular substructure of the mixed graph defined in the previous section.

Given subsets V_1 and V_2 of $V(G)$, each of cardinality r, a nonsingular substructure of G relative to V_1 and V_2 (as described in Section 3) may be called a *generalized matching* between V_1 and V_2. If $V_1 \cap V_2 = \emptyset$, one can easily see that a nonsingular substructure of G relative to V_1 and V_2 is indeed a matching, and a combinatorial interpretation of the corresponding minor of $L(G)$ amounts to the sum of $+1$ or -1 associated with each of these matchings. In this section, we would like to describe this "sign" of a generalized matching using the labeling of vertices and structure of its components.

For a generalized matching $S = (V_1 \cup V_2, F)$ between V_1 and V_2, let $T_1, T_2, \ldots, T_p$ be its tree components and Q be the union of the remaining components on $V_1 \cap V_2$. Let A_1 denote $M[V_1, F]$ and A_2 denote $M[V_2, F]$. Let $V_1 \backslash V_2 = \{u_{10}, u_{20}, \ldots, u_{p0}\}$, where $u_{10} < u_{20} < \cdots < u_{p0}$, and u_{i0} is a vertex of T_i for $i = 1, 2, \ldots, p$. Let $V_2 \backslash V_1$ be given by $\{u_{1t_1}, u_{2t_2}, \ldots, u_{pt_p}\}$ where u_{it_i} is a vertex of T_i for $i = 1, 2, \ldots, p$, and t_i is the length of the unique path γ_i in T_i from u_{i0} to u_{it_i} for $i = 1, 2, \ldots, p$. Let U_i be an ordered list of vertices of $V_1 \cap V_2$ on the path γ_i between u_{i0} and u_{it_i} for $i = 1, 2, \ldots, p$, and let E_i be an ordered list of edges on γ_i for $i = 1, 2, \ldots, p$. Let U and E denote ordered lists of vertices and edges, respectively, of S outside $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_p$.

It is clear that we can order the columns of A_1 and A_2 in the list $E_1, E_2, \ldots, E_p, E$ by using the identical permutation, without changing the value of $\det(A_1)\det(A_2)$. We can order the rows of A_1 in the list $u_{10}, u_{20}, \ldots, u_{p0}, U_1, U_2, \ldots, U_p, U$ to get $\widetilde{A}_1$, and order the rows of A_2 in the list $u_{1t_1}, u_{2t_2}, \ldots, u_{pt_p}, U_1, U_2, \ldots, U_p, U$ to get $\widetilde{A}_2$. Observe that for $i = 1, 2, \ldots, p$ the row u_{i0} has to move past $u_{i0} - i - \alpha_i$ rows, and the row u_{it_i} has to move past $u_{it_i} - i - \beta_i$ rows, where $\alpha_i = |\{j | u_{jt_j} < u_{i0}\}|$ and $\beta_i = |\{j | u_{j0} < u_{it_i}\}|$. Note that $\sum_{i=1}^{p} \alpha_i + \sum_{i=1}^{p} \beta_i = p^2 \equiv p \pmod 2$. Therefore

$$\det(\widetilde{A}_1)\det(\widetilde{A}_2) = (-1)^{\sum_{i=1}^{p}(u_{i0}+u_{it_i})}(-1)^{\mathrm{inv}(\tau)+p}\det(A_1)\det(A_2), \quad (4.1)$$

where τ is an order-preserving bijection between $\{1, 2, \ldots, p\}$ and $\{u_{1t_1}, u_{2t_2}, \ldots, u_{pt_p}\}$ and $\mathrm{inv}(\tau)$ is the number of inversions of τ. If B_1 is obtained from $\widetilde{A}_1$ by pushing all rows of U_i just below the row u_{i0} for $i = 1, 2, \ldots, p$, and B_2 is obtained from $\widetilde{A}_2$ by pushing all rows of U_i just below the row u_{it_i} for $i = 1, 2, \ldots, p$, it is clear that the number of inversions in both "pushing up" movements are identical. Due to the block diagonal structure of both B_1 and B_2, it is easy to calculate

$$\det(B_1)\det(B_2) = (-1)^{p+t_1+t_2+\cdots+t_p+d(\gamma_1\cup\gamma_2\cup\cdots\cup\gamma_p)}\det(M[U,E])^2, \quad (4.2)$$

where $d(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_p)$ denotes the number of oriented edges in $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_p$. From (4.1) and (4.2), we get

$$\det(A_1)\det(A_2) = (-1)^{\sum_{i=1}^{p}(u_{i0}+u_{it_i}+t_i+d(\gamma_i)+1)+\mathrm{inv}(\tau)+p}\det(M[U,E])^2.$$

It is clear that $\det(M[U,E])^2 = 4^{\omega_1(S)}$, where $\omega_1(S)$ is the number of nonsingular unicyclic components of S. We can define the sign of S as

$$\mathrm{sgn}(S) = (-1)^{\sum_{i=1}^{p}(u_{i0}+u_{it_i}+t_i+d(\gamma_i))+\mathrm{inv}(\tau)},$$

where $V_1 \backslash V_2 = \{u_{i0} | 1 \le i \le p\}$, $V_2 \backslash V_1 = \{u_{it_i} | 1 \le i \le p\}$, γ_i is the unique path in T_i from u_{i0} to u_{it_i} of length t_i for $i = 1, 2, \ldots, p$, and τ is the unique order-preserving map from $\{1, 2, \ldots, p\}$ to $\{u_{1t_1}, u_{2t_2}, \ldots, u_{pt_p}\}$ where $u_{10} < u_{20} < \cdots < u_{p0}$.

By using the definition of the sign of a generalized matching, we have the generalized Matrix Tree Theorem:

Theorem 3 *Let G be a mixed graph, and let $V_1, V_2 \subseteq V(G)$ with $|V_1| = |V_2|$. Then*

$$det(L[V_1, V_2]) = \sum_R sgn(R)\, 4^{\omega_1(R)},$$

where the sum is taken over generalized matchings R between V_1 and V_2.

Proof This follows immediately from the discussion before the theorem. ■

Corollary 3 *Let G be a quasi-bipartite graph, and let V_1, $V_2 \subseteq V(G)$ with $|V_1| = |V_2|$. Then*

$$det(L[V_1, V_2]) = \sum_R sgn(R),$$

where the sum is taken over generalized matchings between V_1 and V_2.

COROLLARY 4 *Let G be a mixed graph, and let $V_1, V_2 \subseteq V(G)$ with $|V_1| = |V_2|$ and $V_1 \cap V_2 = \emptyset$. Then*

$$det(L[V_1, V_2]) = \sum_R sgn(R),$$

where the sum is taken over matchings R between V_1 and V_2.

Proof This follows immediately by the characterization of the nonsingular substructures in Theorem 2, as the disjointness of V_1 and V_2 force each component of a nonsingular substructure R to be a tree with exactly one edge connecting a vertex of $V_1 \backslash V_2$ with a vertex of $V_2 \backslash V_1$, *i.e.*, a matching between V_1 and V_2. ■

5. STRUCTURE OF QUASI-BIPARTITE GRAPHS

In this section we take a closer look at the structure of quasi-bipartite graphs. The next result provides alternative definitions of such graphs. A *signature matrix* is a diagonal matrix with ± 1 along the diagonal.

THEOREM 4 *Let G be a mixed graph with incidence matrix M. Then the following conditions are equivalent*:

(i) *G is quasi-bipartite.*
(ii) *There exists a signature matrix D such that the column sums of DM are all 0, where M is the incidence matrix of G.*
(iii) *There exists a signature matrix D such that DLD^t has all off-diagonal entries 0 or -1, where L is the Laplacian matrix of G.*
(iv) *There exists a partition $V(G) = V_1 \cup V_2$ such that every edge <u>between</u> V_1 and V_2 is unoriented and every edge <u>within</u> V_1 or V_2 is oriented.*

Proof We assume that G is connected, since the general result can be obtained by treating the connected components separately.

(i)$\Rightarrow$(ii). We will work with $GF(3)$, the finite field consisting of $\{-1, 0, 1\}$, where the operations are addition and multiplication modulo 3. If G is quasi-bipartite, then M has row-rank (over $GF(3)$) less than ν, the number of vertices in G. Thus there exists a vector $x = (x_1, x_2, \ldots, x_\nu)^t$ over $GF(3)$ such that $x^tM = 0$. It is easy to see that since G is connected, x cannot have a 0 component. Now let D

be the signature matrix with $x_1, x_2, \ldots, x_\nu$ as its diagonal entries. It follows that DM has column sums 0.

(ii)⇒(iii). Let D be a signature matrix such that the column sums of DM are all 0. Then each column of DM must contain a 1 and a -1, the remaining entries being 0. Thus $DMM'D'$ is the Laplacian matrix of a directed graph and therefore it has off-diagonal entries 0 or -1.

(iii)⇒(iv). Let D be a signature matrix such that $DMM'D'$ has off-diagonal entries 0 or -1. Let V_1 (respectively, V_2) be the set of those vertices which correspond to a 1 (respectively, -1) on the diagonal of D. Then it can be seen that any edge connecting a vertex in V_1 and a vertex in V_2 must be unoriented, whereas the remaining edges must all be oriented.

(iv)⇒(i). This assertion is easily proved since any cycle in G must contain an even number (possibly zero) of edges that connect a vertex in V_1 and a vertex in V_2, and by (iv), all such edges are unoriented. Therefore, any cycle in G is singular, and G is quasi-bipartite. ■

THEOREM 5 *Let G be a mixed quasi-bipartite graph. Then the absolute values of all the cofactors of the Laplacian matrix $L(G)$ are equal, and their common absolute value is the number of spanning trees of G.*

Proof The result follows from the equivalence of (i) and (iii) in Theorem 4 and the generalized Matrix Tree Theorem. ■

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